Reading *Begriffsschrift*

By Danielle Macbeth

FREGE'S *BEGRIFFSSCHRIFT* IS UNIVERSALLY REGARDED AS A NOTATIONAL VARIANT of standard quantificational logic: its concavity and German letter is read as a universal quantifier, its two-dimensional conditional stroke as a truth-functional connective, and its Latin italic letters as variables. But while *Begriffsschrift can* be so read, it can also be read very differently, as a notation of a fundamentally different kind of logical language. My aim here is to outline such a reading.[1]

It is a familiar fact that different systems of notation can function in radically different ways. Consider, to take a very simple example, the difference between the sign-designs 'twenty-three', 'XXIII', and '23'. The first is an expression of English tracing the sounds a speaker makes in uttering the words 'twenty' and 'three'. The second is a Roman numeral that uses signs for collections of things—'X' for ten things and 'I' for one thing—to present by addition the idea of ten and ten and one and one and one, that is, twenty-three, things. Instead of tracing the sounds a speaker makes in speaking in some natural language, the Roman numeral 'XXIII' directly represents a collection of things. The Arabic numeral '23' is different again. Like the Roman numeral 'XXIII' it in some way represents a number directly; unlike the Roman numeral it is not immediately additive. The numeral '23' is not to be read as designating two and three things. The Arabic numeration system functions as a notational system in a way that is different both from the notational system of a written natural language such as English and from that of the Roman numeration system.

Begriffsschrift is clearly some sort of written language the aim of which is the expression of a thought. But what sort is it? Three different conceptions are relevant to our purposes here. The first, and most primitive, is a conception of written language as a record of speech enabling one capable of reading the language to reproduce the relevant sequence of phonemes. Sentences in such a language are written sequentially, or serially, left to right, and they are to be read sequentially

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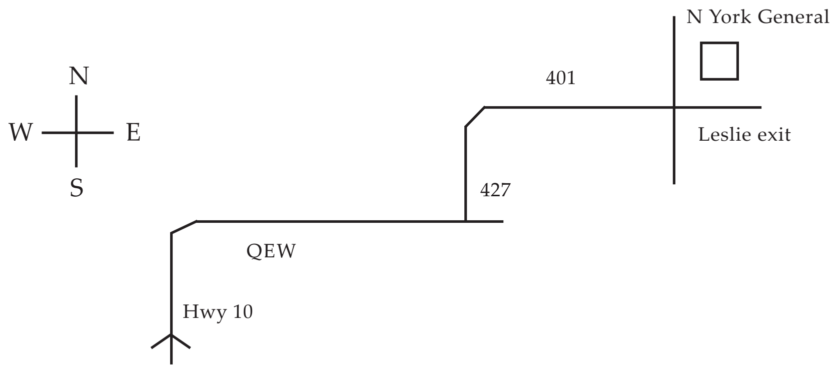
(serially), left to right. They are to be read, that is, much as a route map is to be read, as a series of directions. A sentence of written language so conceived is, as Frege puts it, "a direction for forming a spoken sentence in a language whose sequences of sounds serve as signs for expressing a sense."[2] But, as Frege goes on, although at first the connection between the written word (say, 'Socrates') and that which it signifies (the individual man Socrates) is mediated by the relevant sounds, "once this connection is established, we may also regard the written or printed sentence as an immediate expression of a thought, and so as a sentence in the strict sense of the word." We can learn, that is, to read a sentence of ordinary written language in a new way, to read it as a sentence of a fundamentally different kind of language. Instead of reading a sentence of English as "a direction for forming a spoken sentence," we learn to read it as itself "an immediate expression of a thought." Because the meaning of the language as it is conceived in this second way is carried by the written signs themselves, such a language is "written for the eye" not merely in the trivial sense that it is written but also in the more interesting sense that the meanings of sentences are now seen, as it were, rather than heard.

Consider a route map, for instance, this:



Though we find it easy to take in such a display as a whole, this map was not drawn to represent the relative orientations of landmarks. It was written to be read serially, as a set of directions, in effect, this: take highway 10 north to the QEW east exit then follow the QEW to 427N, go east onto 401 to the Leslie exit, then look for North York General. A route map is in this way both a record of a journey and a set of directions for repeating the journey, not a presentation of the relative positions of various landmarks. Once one *has* drawn such a map, however, it is not hard to learn to read it differently, simultaneously rather than serially, as a two-dimensional map presenting the relative locations of landmarks. Read this second way, the map provides a kind of bird's eye view of the layout of landmarks. Though we can only experience any reasonably large portion of space piecemeal, through our movements from landmark to landmark, we learn in this way to conceive the relevant layout of land as an integrated whole, each of the landmarks within it as having a place relative to all the others. And in much the same way, once it is written we can learn to read a sentence of natural language in a new way, not serially, as a set of directions for forming a spoken sound, but simultaneously, as a kind of picture presenting objects (cognitive landmarks) as thus and so, that is, as presenting a state of affairs, objects with their various properties and relations. So read, a sentence such as 'Romeo loves Juliet' (say) pictures the circumstance of Romeo loving Juliet: 'Romeo' stands in for or represents Romeo, 'Juliet' represents Juliet, and by writing their names as we have—that is, to the left and right respectively of 'loves'—we show that the two people represented stand in the relation of loving, the former to the latter.

Ordinary written natural language can come to be read simultaneously, as a language for the eye. But written natural language was not formed as such a language. As Frege says, "it simply reproduces the verbal speech," and because it does, "there is only an imperfect correspondence between the way the words are concatenated and the structure of the concepts."[3] "Speech [and so also verbal language which reproduces it] often only indicates by inessential marks or by imagery what a concept-script should spell out in full."[4] A concept-script, by contrast with verbal language, is explicitly designed in such a way that "it directly expresses the facts without the intervention of speech."[5] Whereas in English one would write, for instance, 'Romeo loves Juliet', in a more perspicuous notation, in a concept-script, one might write simply 'Lrj' understood as (that is, to be read as) a presentation of two individuals, Romeo and Juliet, in a relation of loving, the former to the latter. Such a notation is logically more perspicuous than written English in two ways. First, it is not mediated by spoken English; one cannot read 'Lrj' as one can read written English, as a direction for forming a spoken sentence—though of course one can translate it into such a sentence. The letter 'r' in such a language stands for Romeo not by way of the sound the letter makes when it is spoken but directly. It is a simple sign to be understood as representative of Romeo. The sentential sign 'Lrj' is also more perspicuous than written English insofar as it does not, as written English does, mark one object name as designating the subject. In English, one distinguishes between 'Romeo loves Juliet' and 'Juliet is loved by Romeo'; Romeo is the grammatical subject of the first sentence, that to which the hearer's attention is to be directed, and Juliet is similarly the grammatical subject of the second sentence. In standard logical notations, as in *Begriffsschrift*, such a distinction is not marked. 'Lrj' is to be read neither left to right nor right to left but is instead to be taken in as a whole. It provides in this way a kind of bird's eye view of a state of affairs.

Given a map, that is, a presentation of landmarks in their relative positions, it is easy to recover various routes through it. Although the map itself merely presents landmarks in their relative positions, one can nevertheless read it serially in different ways, and use it in order to discover, for instance, the simplest, or shortest, route from one landmark to another. Similarly, although a sentence such as 'Lrj' is itself merely a presentation of objects as thus and so, one can nevertheless read it serially in different ways, take any of a variety of routes through the sentence. One can, for instance, see the sentence 'Lrj' as about Juliet that she is loved by Romeo, a reading that is required, for example, in the context of an inference from the claim that anyone loved by Romeo is happy to the conclusion that Juliet is happy. That Juliet has the property *loved by Romeo* is not itself (directly) depicted in the sentence 'Lrj' any more than a map itself (directly) depicts that in, say, driving from Philadelphia to Pittsburgh on I-76 one will go through Harrisburg. One can read the sentence, or map, that way, but in itself, independent of such a context of use, both a sentence and a map on our second conception merely present things as thus and so.

On our second conception of it, a map uses signs for landmarks to present those landmarks as spatially related in various ways. What is depicted in a map, on this conception, is objects (landmarks) in their various relations as if seen from above. Given such a conception, we can learn to read a map in a different way again, as a presentation of a space as an antecedently given whole (that is, as a whole that is prior to its parts) within which landmarks are directly located, each independently of all the others. Whereas on the second conception one begins with objects (landmarks) and then locates them relative to one another, on this third conception one begins not with things but with space itself abstractly conceived as a given, irreducible whole laid out as a grid within which individual objects can be directly located. Space so conceived is clearly intelligible prior to, and so independent of, any reference to objects. The relative locations of objects, while they can be read off of a map of this sort, are not a given of the map itself as they are on the second conception. What is presented in a map on this third conception is not a bird's eye view of things but instead a specification of the locations of objects directly in an antecedently given expanse of space—as it were, by their Cartesian coordinates.

In the case of maps, on the second conception of them, we begin with signs for landmarks and show the relative positions of those landmarks by a relative placement of the signs for them. Similarly, in written language on the second conception of it, we begin with signs representative of objects and show their properties and relations one to another in sentences such as 'Lrj'. In written language so conceived, the primitive signs 'r' and 'j' are taken to designate objects prior to and so independent of their occurrence in sentences. But just as, on our third view of it, we can conceive of (a portion of) space as a given whole that can be carved up into particular spaces (into, say, roads and intersections) in a variety of ways, so in the case of written language, we can learn to conceive a portion of it, a sentence, as a kind of a given whole that can be carved up into particular subsentential expressions (object names and concept words) in a variety of ways. What the sentence *itself* pictures or maps, on this conception, is not objects with their various properties and relations, but only a sense, a Fregean thought—that is, as we might think of it, a (cognitive) space, but not yet particular spaces where objects, properties, and relations might be found. Relative to an analysis into function and argument, the subsentential parts of the sentence can be understood referentially, that is, as designating objects or concepts; but independent of any analysis the sentence is to be understood to express only a thought, and, if there is one, to designate only a truth-value, either the True or the False.

According to Frege, one of the most important and fundamental insights codified in his logical language is that the subject/predicate distinction is to be replaced by the distinction of argument and function, where argument and function are given only relative to an analysis.[6] "In this," he claims in *Begriffsschrift* §3, "I strictly follow the example of the formula language of mathematics, in which, also, one can distinguish subject and predicate only by doing violence." The point is developed for the case of the sentence '$2^4 = 16$' of the formula language of arithmetic in the long Boole essay written shortly after *Begriffsschrift*.

> If . . . you imagine the 2 in the content of possible judgment
>
> $$2^4 = 16$$
>
> to be replaced by something else, by (-2) or by 3 say, which may be indicated by putting an x in place of the 2:
>
> $$x^4 = 16$$
>
> the content of possible judgment is thus split into a constant and a variable part. The former, regarded in its own right but holding a place open for the latter, gives the concept '4th root of 16'.
>
> We may now express
>
> $$2^4 = 16$$
>
> by the sentence '2 is a fourth root of 16' or 'the individual 2 falls under the concept "4th root of 16"' or 'belongs to the class of 4th roots of 16'. But we may also just as well say '4 is a logarithm of 16 to the base 2'. Here the 4 is being treated as replaceable and so we get the concept 'logarithm of 16 to the base 2':
>
> $$2^x = 16$$
>
> The x indicates here the place to be occupied by the sign for the individual falling under the concept.[7]

Although we perhaps find it most natural to read the sentence '$2^4 = 16$' of the formula language of arithmetic as built up from the antecedently meaningful parts '2', '4', and '16', each of which functions to refer to a particular number, arranged so as to express the arithmetical fact that two raised to the fourth power is equal to sixteen, Frege suggests that we can read the sentence differently. On his reading, a numeral such as '2' seems to designate the number two only relative to a certain analysis. More generally, on this reading, meaning (*Bedeutung*) can be assigned to the subsentential parts of a sentence only relative to an analysis into function and argument.[8]

The written language, as we now conceive it, is a given whole articulating a (metaphorical) space of possible contents. The various primitive expressions of the language express senses independent of the context of a proposition—which explains, by compositionality, our ability to grasp the thoughts expressed by sentences never before encountered[9] —but independent of a use, real or imagined, they are not to be taken to designate anything. In the formula language of arithmetic, for instance, we can form an equation such as '1+1+1 = 3' which both expresses a thought and signifies a truth-value. Then, by analysis, we can form object names, for example, the object names '1+1+1' and '3' which designate the arguments for the two-place relation $\xi = \zeta$, or alternatively the object name '1+1' which designates the argument for the concept $\xi+1 = 3$.[10] Similarly, while we *can* read a sentence such as 'Cato killed Cato', or 'Kcc', as presenting a state of affairs, each occurrence of 'Cato' as a sign representative of Cato and the whole exhibiting a relation Cato bears to himself, we can also learn to read it differently, as the expression of a Fregean thought, one that can be analyzed into function and argument in various ways, no one of which is privileged. The first guiding principle of our reading is that Frege's logical language is a language of this third sort. Sentences of this language do not serve as a direction for forming a spoken sentence, nor do they present things (antecedently given) as thus and so. Instead they express senses. Only relative to an analysis into function and argument are objects and concepts designated by the subsentential expressions of the language.

Our second guiding principle concerns expressions of generality in the language, in the simplest case, sentences that in English are of the form 'all S is P' and on a quantificational conception are taken to express facts about objects, that all objects (in the domain of quantification) are P if S. In his *Outlines of Scepticism*, Sextus Empiricus contrasts such a conception of generality with another that is much closer to what is wanted here. Suppose that it is true that all Fs are G, whether necessarily, as a matter of law, or merely by accident. Obviously, then, if it is given that the object o is F, one can legitimately infer that o is G. The argument form:

All F is G.
o is F.
Therefore, o is G.

is valid. Nevertheless, Sextus suggests, such an argument form can be conceived in either of two very different ways; for, he argues, depending on the status of the generality, the inference either is circular or has a redundant premise. His example is the generality 'everything human is an animal', and we are to assume, first, that the generality 'everything human is an animal' is true merely by coincidence, in virtue of the fact that as it happens each and every human is also an animal. In that case, Sextus claims, the inference

Everything human is an animal.
Socrates is human.
Therefore, Socrates is an animal.

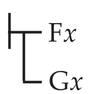
is circular because the fact that Socrates is an animal "is actually confirmatory of the universal proposition in virtue of the inductive mode."[11] Because in the case of this merely contingent or matter of factual generality, Sextus argues, one cannot establish that everything human is an animal without first establishing that Socrates, one of the humans, is an animal, the first premise presupposes already the conclusion; the argument (Sextus claims) is circular. If, on the other hand, we assume that the generality is lawful, that is, that "being an animal follows being human . . . then at the same time it is said that Socrates is human, it may be concluded that he is an animal . . . and the proposition 'Everything human is an animal' is redundant."[12] In the case in which there is a lawful connection between being human and being an animal, Sextus thinks, the argument is valid even without the proposition 'everything human is an animal'. 'Socrates is human; therefore, Socrates is an animal' is not, that is, enthymematic on his view of this case; it is valid just as it stands—though not formally valid. Clearly, then, Sextus does not understand the law as it figures in this case as a kind of necessary truth; were it such a truth, one could not validly infer that Socrates is an animal solely on the basis of the claim that Socrates is human. What Sextus seems to think instead is that the law serves as a kind of inference license, as a kind of rule, that it functions (as the point might be put) not as a claim *from which* one reasons but instead as a principle or rule *according to which* one reasons.[13] Because what such a principle or rule licenses just is one's concluding that Socrates, say, is an animal given that he is human, it would be inconsistent with its status as such a license were one to require its inclusion among the premises. That is just Sextus's point. One *can* include the license among one's premises (in order, perhaps, to make as explicit as possible the modes of inference employed in the proof), but one need not; and if one does, one is not transforming an invalid argument into a valid argument.[14]

Whether or not Sextus is right to think that in the case in which the generality is merely contingent the argument is inherently circular, his analysis clearly points to two very different conceptions of the logical structure of the argument. On the first conception, the thought expressed by the sentence 'everything human is an animal' is understood as something *from which* to reason, as a fact (assuming it is true) that can supply a premise for an argument. On the second conception, the thought expressed by that same sentence is understood not as a claim but as an inference license, as something *according to which* one reasons. Suppose, now, that despite this intuitive difference between the two sorts of inference, the difference between the two sorts of generalities they involve is to be taken to concern not the contents expressed but instead the sort of justification that is involved in grounding their truth. Clearly there are two options: take the first case, that of a contingent generality, as one's paradigm and formulate a plausible account for that case, then treat the second, lawful, case the same way; or take the second, lawful, case to be paradigmatic, give a plausible account of that case, then treat the accidental cases similarly.

According to the first strategy, which begins with the case of contingent generalities, the task is to articulate an understanding of a sentence of the form 'all A is B' where the truth of the sentence is merely a matter of contingent historical fact. What in that case should we say that 'all A is B' means? The most plausible answer is simply this: what is the case if it is true. That is, much as it is natural to understand the sentence 'Socrates is snub-nosed' in terms of its truth-conditions, in terms of the circumstance that obtains if it is true, so it is natural to understand a contingent generality such as, say, 'all Greek philosophers are snub-nosed' in terms of its truth-conditions, what is the case if it is true, namely, that everything that is a Greek philosopher is also snub-nosed. That, of course, is not true. But it could have been true, and what the generality says on this account is simply what is the case on the assumption that it is true.

Extending the account to the case of lawful generalities is straightforward. Because any law (whether conceived as a necessary truth or conceived as a rule of inference) entails a corresponding fact, it is the corresponding fact that we express in our language. Suppose, for instance, that it is a law that humans are mortal (that is, as Sextus would put it, that being mortal follows being human). It follows that each and every human is mortal. What the relevant sentence of our language says is what is the case if that is true, namely that everything falling under the concept *human* falls also under the concept *mortal*. The law is thereby reduced to a fact about objects falling under the relevant concepts. The guiding idea of the strategy—that for the purposes of logic a lawful generality can be treated as a generality that is true just in case things are a certain way—is grounded in an insight: whether it is accidentally true or whether it is a law that all As are B, the facts remain the same.

The second strategy begins with the lawful case, and here the most natural account, following Sextus, is not in terms of the truth-conditions of claims but instead in terms of the authority of rules to license judgments on the basis of other judgments. What is wanted is a form of expression that is to be understood not as a statement of a (necessary) truth but as a statement of an inference license, of something according to which to reason. We could, for instance, introduce the form:



with the stipulation that it means not that anything that is G is or must be F but instead that being F follows being G, that is, that it is permitted to judge that o is F, for some object o, if it is known to be true that o is G. The sentence so conceived does not say that everything falling under the concept G falls or must fall also under the concept F; it does not say how things are concerning any objects at all. What it expresses is a rule, an inference license, something according to which to reason. Of course, if it is a *valid* inference license, one that ought to govern one's reasoning, then it follows that '$(\forall x)(Gx \supset Fx)$' is true, that is, that what this sentence of quantificational logic says is so; and contrariwise, if something is found that is G but not also F, then that is enough to show that the inference license is not valid and should not be adopted as a rule according to which to reason. Nevertheless, what the rule expresses is not the claim that all Gs are F but only a rule licensing certain conclusions on the basis of relevant premises.

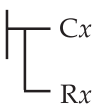
Now we extend the account to the case of accidental generalities. Suppose that it has been established as a matter of contingent empirical fact that all the Gs there are are F. Our strategy demands that we express this claim as an inference license to the effect that one is permitted to infer from the acknowledged fact that o is G (for some object o) that it is F. Because inferences from premises whose truth is not explicitly acknowledged are never permissible—at least on Frege's mature view[15] —treating the claim as an inference license is unproblematic. In counterfactual cases the best one will be able to do is to infer, for some object o that is not in fact G, that if o is G then it is F. Because the antecedent is false in that case, the conditional is true. This strategy, too, is grounded in an insight. Whether it is accidentally true or instead a law that all As are B, the inference potential of the sentence is the same: in either case it is permissible, given that one knows that an object o is A, to infer that o is B. According to the reading pursued here, it is this second strategy that Frege adopts. On this reading, a *Begriffsschrift* generality of the form:

is to be understood as the expression of an inference license, as the expression of a rule according to which to reason rather than as a claim regarding what is, or must be, the case.

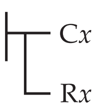
Two guiding principles of our reading have been outlined. First Frege's logical language is to be read as directly enabling the expression of sense; only relative to an analysis do subsentential expressions of the language designate objects and concepts. The second principle is that genuine hypotheticals of *Begriffsschrift*, that is, generalized conditionals expressed using Frege's Latin italic letters, are to be understood as serving in the expression of rules of inference according to which to reason as they contrast with claims from which to reason. The reading itself will be developed in stages.

We begin with a very primitive language, Sellars's 'Jumblese', a language containing only object names.[16] In Jumblese, instead of using predicate expressions to ascribe properties and relations to the objects named one writes those names themselves in various ways. To depict something, the object o, as red, say, one writes its name in (say) bold: **o**. To depict Romeo and Juliet in the relation of loving one perhaps writes a name for the former just before a name for the latter: rj. In this way, in Jumblese, one exhibits how things stand with objects by exhibiting names, representatives of those objects, in various ways. In such a language one does not say how things are; rather one shows how things are. And one shows that two things have something in common, that they share some feature, by writing their names the same way. Now we enrich the language slightly with the introduction of signs for properties and relations. There are two essentially different ways to understand what such an addition achieves. First, we can think of these new letters as merely a notational convenience, as, for instance, Wittgenstein seems to in the *Tractatus* (see §3.1432). Even in his early writings Frege indicates that his own view is that signs for properties and relations, like signs for objects, are representatives. At first he takes them to be representatives of properties and relations; after the early 1890s he takes them to be representatives of senses, to designate concepts—that is, laws of correlation from objects to truth-values—only relative to an analysis. We here follow Frege's later usage. Sentential signs such as 'Ro' and 'Lrj' are to be read as presentations of the senses of sentences that can be analyzed in various ways into function and argument.

In the language as it has been developed to this point, as in natural language, "logical relations are almost always only hinted at—left to guessing, not actually expressed."[17] The next step is to make these relations explicit. We need, that is, to be able to express general laws about concepts, which requires in turn our moving up a level through the development of a sign for the conditional and the introduction of the literal notation. It is, for instance, always in order to judge of an object that it is (say) colored given that it is red. What we want to show in our language is that this inference is a good one whatever the object being considered. We do so by, first, showing that the judgment of some particular object that it is colored can be grounded in the judgment of it that it is red through the use of the conditional stroke, then replacing the object names with Latin italic letters, or indeed any sort of squiggles so long as they are equiform, like this:



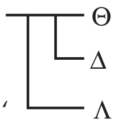
This sentence shows, on our reading, that its being red is a sufficient condition for the judgment of a thing that it is colored, that is, that one can judge of something that it is colored on the basis of the judgment that it is red; and it does so by presenting one concept subordinate to another. Through its use of Latin italic letters lending generality of content and the two-dimensional conditional stroke, the sentential sign:



exhibits the properties of being red and being colored in the logical relation of subordination, and it does so in a way that is strictly analogous, at a higher level, to the way the sentence 'rj' of Jumblese exhibits Romeo and Juliet in the relation of loving. But whereas the Jumblese sentence 'rj' exhibits a relation among objects, a *Begriffsschrift* generalized conditional exhibits a relation among concepts, and because it does, this sentence (unlike 'rj') shows thereby that certain inferences are legitimate. It shows that a judgment of a thing that it is colored is sufficiently grounded on the judgment that that thing is red, that one can judge of something that it is colored on the basis of the judgment that it is red. It is important that we understand just how this is to work.

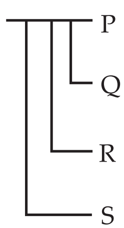
First, as Frege himself emphasizes, Latin italic letters of *Begriffsschrift* fundamentally contrast with all other signs of that language. As the point is put in the late fragment "Logical Generality," such letters do not have the form of words.[18] They do not designate anything (even indeterminately); and they do not express a sense.[19] That is why we could have used any sort of meaningless squiggle in place of Frege's Latin italic letter. The role of Latin italic letters in the language is *only* to lend generality of content. It is to enable one to show that concepts stand in logical relations (paradigmatically the relation of subordination), and thereby to show that certain transitions in the language are good ones whatever the objects under consideration.

But why the two-dimensional conditional stroke? Why not a one-dimensional sign such as the horseshoe of standard notation? The rationale is essentially that given above for the case of a simple singular sentence such as 'Lrj' or '$2^4 = 16$'. While a multiply embedded conditional is *constructed* in a stepwise fashion, in a process that is codified in standard one-dimensional notations through the use of brackets, we can learn to read the resultant sentence differently, as merely exhibiting sentences in logical relations. So read, such sentences are variously analyzable. As Frege himself points out in *Grundgesetze* (§12), multiply embedded conditional sentences of *Begriffsschrift* have a main connective only relative to an analysis:

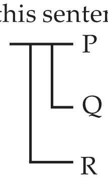
> In '  ' we may call '—Θ' the *main component* and '—Δ' and '—Λ' *subcomponents*; however, we may also regard '  ' as the *main component* and '—Λ' alone as *subcomponent*.

One can, in other words, take various different perspectives on a sentence with this sort of logical structure in Frege's *Begriffsschrift*; one can analyze it in different ways.

Consider, for example, the *Begriffsschrift* sentence:



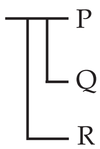
If '—S' in this sentence is treated as the subcomponent and



as the main component, whose main component in turn is

then the result is what would be most naturally expressed in quantificational logic as ‘S⊃(R⊃(Q⊃P))’.

Alternatively, treating ‘—S’ again as subcomponent and

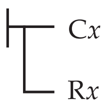


as main component but now taking the main component of this component to be ‘—P’, leaving ‘—R’ and ‘—Q’ as subcomponents, yields ‘S⊃((R&Q)⊃P)’.[20] If one instead treats ‘—S’ and ‘—R’ as subcomponents and:

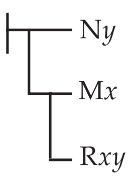
as the main component, then the result might be expressed in standard notation as ‘(S&R)⊃(Q⊃P)’. If, finally, all of ‘—S’, ‘—R’ and ‘—Q’ are treated as subcomponents with ‘—P’ as the main component, we get ‘(S&R&Q)⊃P’. Each of ‘S⊃(R⊃(Q⊃P))’, ‘S⊃((R&Q)⊃P)’, ‘(S&R)⊃(Q⊃P)’, and ‘(S&R&Q)⊃P’ represents in this way one path through Frege’s two-dimensional structure, one way to think about its constructional history. Though the equivalence of these four formulae must be proven in a standard one-dimensional notation, it is a given of Frege’s two-dimensional notation. As far as its inference potential is concerned, Frege’s one formula corresponds to an equivalence class of formulae in standard one-dimensional notation. The only case in which the two are comparable is the limit case of a simple conditional; only in this case is there a one-to-one correspondence of the *Begriffsschrift* conditional and the conditional as it is normally expressed.

In standard notation, on the standard reading of it, there is always a main connective; one is to understand a sentence such as ‘P⊃(Q⊃R)’ as a conditional whose antecedent is ‘P’ and whose consequent is ‘Q⊃R’. The sentence in this way codifies its constructional history much as ‘$2^4 = 16$’ on our first reading of it codifies its constructional history. And just as ‘$2^4 = 16$’ itself says, on our first reading of it, that two to the fourth power is equal to sixteen, so the sentence ‘P⊃(Q⊃R)’, as it is usually read, itself says that either ‘P’ is false or ‘Q⊃R’ is true. The comparable thought expressed in Frege’s notation does not in the same way say that either ‘P’ is false or ‘Q⊃R’ is true. Instead it exhibits a logically complex relationship among the three sentences P, Q, and R, one that can be analyzed as saying that ‘P’ is false or ‘if Q then R’ true but can be analyzed in other ways as well. Once we see this it is easy to see, in Frege’s notation, that interchanging subcomponents is permissible. Changing the order of subcomponents does need to be justified, which is why interchange of subcomponents is given as a rule in *Grundgesetze* and is proved as a theorem in *Begriffsschrift*; but in Frege's notation one such rule can cover all cases of this form of embedding. In our standard notations, by contrast, having proven that (say) '(P&Q)⊃R' is equivalent to '(Q&P)⊃R' will not save one the trouble of having also to prove that (say) 'P⊃((Q&R)⊃S)' is equivalent to 'Q⊃((R&P)⊃S)'. Indeed these look to be quite different sorts of cases in standard notation. Where Frege has one rule and a two-dimensional notation to fix the equivalence of the four (linear) sentences considered above as well as all twenty variants with 'S', 'R', and 'Q' in different orders, a standard linear notation brings with it the demand that one prove, for each pair of the twenty-four sentences involved, that they are equivalent. Frege's two-dimensional notation, which can seem pointless and perverse when read as a notation aimed at tracing the step-wise construction of truth-conditions, is especially perspicuous, much more so than a linear notation, when it is read as a notation aimed at the presentation of the inference potential that is common to such logically equivalent sentences. Frege's two-dimensional conditional stroke combines in this way "maximal logical precision, together with perspicuity and brevity."[21]

In virtue of its two-dimensional conditional stroke and the logical role played by Frege's Latin italic letters in lending generality of content, *Begriffsschrift* generalized conditionals exhibit concepts in logical relations of arbitrary complexity. A simple general sentence such as:



exhibits concepts in a logical relation, and in just the same way a sentence such as:



exhibits concepts in a logical relation. All such sentences function in a way that is strictly analogous to the way sentences of our original language Jumblese function, but at a higher level. Whereas the Jumblese sentence 'rj' exhibits a relation among objects, a *Begriffsschrift* generalized conditional expressed using Latin italic letters exhibits a relation among concepts, one that can be analyzed in various ways for the purposes of judgment and inference.

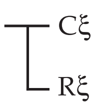
The next step is to introduce signs for the relations left unmarked in *Begriffsschrift* generalized conditionals expressed using Latin italic letters. Frege's concavity notation in combination with the conditional stroke can be read as serving just this purpose. If it is, then the sentence:

is to be read as saying of the concepts *red* and *colored* that the former is related to the latter by the relation of subordination, where this latter (second-level) relation is designated by the expression that is left over when the two concept words, 'Cξ' and 'Rξ', are removed. By contrast with the generalized conditional written using Frege's Latin italic letters, which (on our reading) functions as a rule licensing inferences, such a sentence is to be read as having the form of a claim. So read, it is analogous to 'Lrj' but at a higher level; it stands to its counterpart written using Latin italic letters as 'Lrj' stands to the Jumblese sentence 'rj'. And here again, we can learn to read the sentence in a new way, not as ascribing the second-level relation *subordination* to the first-level concepts *red* and *colored*, but as merely exhibiting the senses of the expressions for such concepts and relations. The sentence so read can be variously analyzed. We can, for instance, take the judgment:

to involve the second-level relation of subordination, designated by the expression:

for arguments Cξ and Rξ, but we can equally well take it to involve the second-level concept:

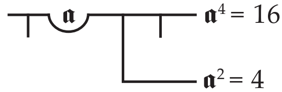
for argument:



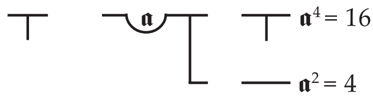
or to involve the second-level concept:

for argument Cξ, and so on.

A particular affirmative such as:

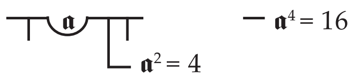


is essentially similar, though owing to its greater logical complexity even more analyses are possible. Depending on how the horizontals are taken to be amalgamated, it can be read, for instance, as follows. As:



it is the judgment that it is not the case that the concepts *square root of four* and *not a fourth root of sixteen* are related by the (second-level) relation of subordination. As:

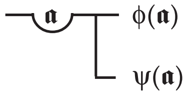
it is the judgment that there is something that is both a square root of four and a fourth root of sixteen. As:



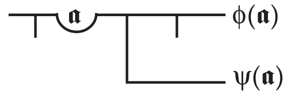
it reads as the judgment that the property *fourth root of sixteen* has the (higher-level) property *property of some (at least one) square root of four* (that is, it is the judgment of some square root of four that it is a fourth root of sixteen). As:

that same sentence ascribes the second-level (logical) property of compossibility to the concepts *square root of four* and *fourth root of sixteen*. On this analysis it says that it is possible for a square root of four to be a fourth root of sixteen.[22] And other analyses are possible as well. Independent of an analysis into function and argument, this particular affirmative of *Begriffsschrift* does not "say" any one of these things to the exclusion of the others. Rather it exhibits the inference potential, the *begrifflicher Inhalt*, that is common to them all. We know that to understand the goodness of inferences involving logical generality sentences must be variously analyzable. In virtue of their two-dimensionality, sentences of *Begriffsschrift*, as understood here, wear this potential on their face.

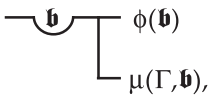
We can also now see how more complex second-level concept words are formed. We have, as our basic cases, the signs:



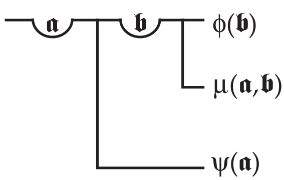
and



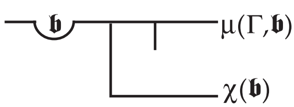
Since only the first can be used in the formation of a law—in the expression of what Frege calls a genuine hypothetical—we begin with it, the second-level relation of subordination, which provides, according to Frege, the fundamental form "of all laws of nature and of all causal connections in general."[23] If we now replace $—\phi(\Gamma)$ in our sign for subordination with the function:



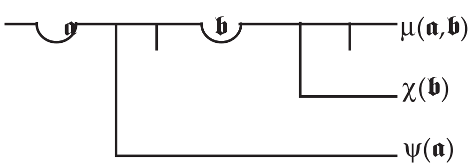
another, more complex, relation is formed:



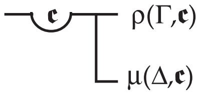
If we further stipulate that $\psi(\xi) = \phi(\xi)$, then this expression designates the relation of following in a sequence. It is correctly ascribed to a two-place (first-level) relation $R\xi,\zeta$ and a (first-level) concept $F\xi$ just if F follows in the R-sequence. If $—\phi(\Gamma)$ is replaced instead with the function:



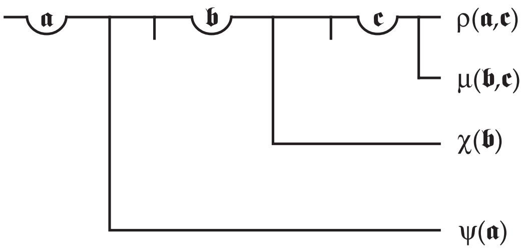
we get:



which is the second-level relation that is ascribed, for instance, to the concepts *boy*, *girl*, and *loves* in 'every boy loves some girl'. Where $\chi(\xi) = \psi(\xi)$, another important second-level relation is designated, that which holds, for instance, of the concept *number* and the successor relation (since every number has a successor). And still more logically complex second-level relations can be formulated as well. If, for instance, the expression '$—\mu(\Gamma,\Delta)$' above is replaced by:



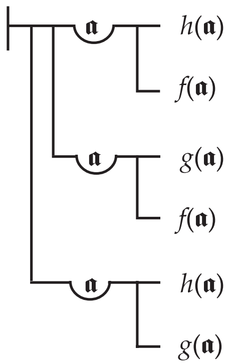
then the resultant concept is:



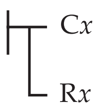
It is this concept that is critical to limit operations. To form, for instance, the second-level unequalled-level relation of continuity which takes a function *f* and a point *A* as arguments, one puts the concept $\xi>0$ for $\psi(\xi)$ and for $\chi(\xi)$, the relation $-\xi \leq \zeta \leq \xi$ for $\mu(\xi,\zeta)$, and the relation $-\xi \leq f(A+\zeta) \leq \xi$ for $\rho(\xi,\zeta)$. The result is a more determinate concept, but since it takes functions and points as arguments, it is one that is nonetheless second-level. Obviously, we could go on.

As these various examples show, first-level concepts and relations can have a variety of different second-level properties and can stand in a variety of different second-level relations, from the relatively simple relation of subordination to the quite complex relations involved in limit operations. Frege's *Begriffsschrift* enables us to easily form expressions designating these second-level concepts and relations. It is these second-level concepts and relations in turn that are the subject-matter of logic as Frege understands it, and only one last step in the development of our language is needed to show how and why that is.

We saw that the logical relationship between a simple sentence such as 'Ro' and another 'Co' can be laid bare in the judgment that *red* is subordinate to *colored* as expressed using Frege's Latin italic letters and the two-dimensional conditional stroke. The sentence '*red* is subordinate to *colored*' similarly stands in certain logical relations to other sentences. One can correctly argue, for instance, that since *red* is subordinate to *colored* and *colored* is subordinate to *extended*, it follows that *red* is subordinate to *extended*. The final level is reached with the use of the literal notation to show that such an inference is a good one no matter what first-level concepts are being considered. The relation of subordination, which is a second-level relation of first-level concepts, is transitive, and we show that this is so by replacing all first-level concept words with Latin italic letters (equiform squiggles), for instance this way:



This sentence, we should now be able to see, exhibits not how things stand with objects, nor even how things stand with (first-level) concepts, but instead how things stand with (second-level) relations of (first-level) concepts. It is analogous to the judgment:



but at a higher level. In virtue of the peculiar expressive capacities of Frege's German and Latin italic letters, this *Begriffsschrift* sentence shows that the second-level logical relation of subordination is transitive. It *exhibits* this general law about concepts and makes essential use of the different expressive capacities of Frege's Latin italic and German letters in doing so. In this sentence, the German letters (together with the concavity and conditional stroke) serve in the formation of a concept name for the relation *subordination*, and the italic Latin letters enable one to show something about this designated concept, namely, that it is transitive. We are thus using a third-level concept, *transitive*, but have no sign for this concept. We show but do not say that a certain second-level concept, *subordination*, which is designated, is transitive.

The laws of logic as expressed in Frege's notation as we read it are then fully contentful truths, albeit higher order. They are laws governing such second-level properties and relations as *subordination*, *being hereditary in a sequence*, and *following (or preceding) in a sequence*; in *Begriffsschrift* Parts II and III, Frege proves that these second-level concepts and relations have various higher order properties and relations, that *subordination* and *following in a sequence* are transitive, and so on. Logic, on Frege's view, is not, then, "unrestrictedly formal." It *is* formal in the sense that "as far as logic itself is concerned, each object is as good as any other, and each concept of the first level as good as any other and can be replaced by it," but logic nonetheless has its own content: "just as the concept *point* belongs to geometry, so logic, too, has its own concepts and relations; and it is only in virtue of this that it can have a content. Towards what is thus proper to it, its relation is not at all formal. . . . To logic, for example, there belong the following: negation, identity, subsumption, subordination of concepts. And here logic brooks no replacement."[24] The concern of the science of logic, on Frege's view, is the higher order concepts and relations under which the (first-level) concepts and relations of the special sciences fall. Because it is, it falls to logic to discover the laws of truth, more exactly, the laws of "that sort of truth which it is the aim of science to discover."[25]

On the reading we have outlined, Frege's *Begriffsschrift* is quite different from the quantificational languages with which we are familiar. His concavity is not a universal quantifier. Though given one sort of analysis it can be taken to designate something very much like our universal quantifier, Frege's concavity is a sign that together with his conditional and negation strokes can be used to form second-level concept words of arbitrary complexity. Frege's two-dimensional conditional stroke, similarly, is not read here as a truth-functional connective. Like concavity, it functions in combination with other signs (other conditional strokes, signs for negation, and the concavity with German letter) to form signs of arbitrary complexity, enabling the presentation of thoughts that can be variously analyzed into function and argument. Frege's Latin italic letters, finally, do not function as (free) variables on our reading. Rather they serve to move everything up a level. Replacing object names with Latin italic letters moves one up from consideration of properties and relations of objects to consideration of (second-level) properties and relations of first-level concepts, and replacing (first-level) concept words with Latin italic letters moves one up again, from consideration of (second-level) properties and relations of first-level concepts to consideration of (third-level) properties and relations of second-level concepts. Frege's *Begriffsschrift*, though it can be regarded as a notational variant of standard quantificational logic, *can also* be read as we have read it here, as a fundamentally different kind of logical language. Is *Begriffsschrift* a notational variant of standard quantificational logic? We simply do not know. φ

Notes

[1] This reading is developed and defended in much greater detail in my *Frege's Logic* (Cambridge, MA: Harvard University Press, forthcoming).

[2] Gottlob Frege, "Logical Generality," in *Posthumous Writings*, ed. Hans Hermes, Friedrich Kambartel, and Friedrich Kaulbach, trans. Peter Long and Roger White (Chicago: University of Chicago Press, 1979), p. 260.

[3] "Boole's Logical Calculus and the Concept-script," in *Posthumous Writings*, pp. 12-13.

[4] "Boole's Logical Calculus," *Posthumous Writings*, p. 13.

[5] Gottlob Frege, "On the Scientific Justification of a Conceptual Notation," in *Conceptual Notation and Related Articles*, trans. and ed. T. W. Bynum (Oxford: Clarendon Press, 1972), p. 88.

[6] See, for instance, the final paragraphs of the preface to Frege's 1879 *Begriffsschrift*, and also §9 of that work, in *Conceptual Notation*, pp. 107, 126.

[7] "Boole's Logical Calculus," *Posthumous Writings*, pp. 16-17.

[8] Frege writes in the long Boole essay, "instead of putting a judgment together out of an individual as subject and an already previously formed concept as predicate, we do the opposite and arrive at a concept by splitting up the content of possible judgment"

("Boole's Logical Calculus," *Posthumous Writings*, p. 17). The point is made again in "On Concept and Object": "a thought can be split up in many ways, so that now one thing, now another, appears as subject or predicate. The thought itself does not yet determine what is to be regarded as the subject" (in *Posthumous Writings*, p. 107). In *Grundlagen*, the same idea appears as the familiar context principle: "never to ask for the meaning of a name in isolation, but only in the context of a proposition" (*The Foundations of Arithmetic*, trans. J. L. Austin [Evanston, IL: Northwestern University Press, 1980], p. x.) It is made again in the 1914 "Notes for Ludwig Darmstaedter": "I do not begin with concepts and put them together to form a thought or judgment; I come by the parts of a thought by analyzing the thought" (in *Posthumous Writings*, p. 253).

[9] It is worth noting that Frege discussed the compositionality of language only after he had introduced the distinction between sense and meaning in the early 1890s. See *Posthumous Writings*, p. 225; and Gottlob Frege, *Collected Papers on Mathematics, Logic, and Philosophy*, ed. Brian McGuinness, trans. Max Black et al. (Oxford: Basil Blackwell, 1984), p. 390.

[10] Frege argues in *Grundlagen* that, given that there is only one one, it is incoherent to read an expression such as '1+1+1' as designating three ones that are then to be put together. There is only one one and yet 1+1+1 = 3, and the way we are to understand this is in terms of the idea that '1+1+1' designates not one and one and one but instead the number three. Only relative to a particular analysis—for example, that according to which the first occurrence of '1' designates the argument and the remainder, 'x+1+1', an arithmetical function—does an occurrence of '1', in the expression '1+1+1', designate the number one.

[11] *Sextus Empiricus: Outlines of Scepticism*, trans. Julia Annas and Jonathan Barnes (Cambridge: Cambridge University Press, 1994), bk. II, §195.

[12] *Outlines of Scepticism*, bk. II, §165.

[13] The formulation is Mill's in *A System of Logic Ratiocinative and Inductive*, 8th ed., ed. J. M. Robson (Toronto: University of Toronto Press, 1973), bk. II, chap. iii, §4.

[14] See Gilbert Ryle's "'If', 'So', and 'Because'," in *Philosophical Analysis*, ed. Max Black (Ithaca, NY: Cornell University Press, 1950), for a more recent discussion of the idea that a generalized conditional should be understood as an inference license.

[15] Frege at first thought that inferences can be drawn from false, or merely unjudged, premises. We are told, for instance, in *Begriffsschrift* §2 that one can derive conclusions from a sentence lacking the judgment stroke and might do so in order to "test the correctness of the thought." In his later writings, Frege again and again emphasizes that "only true thoughts are admissible premises of inferences" (*Posthumous Writings*, p. 180), "only a thought recognized as true can be made the premise of an inference" (*Posthumous Writings*, p. 261), "from false premises nothing at all can be concluded" (*Philosophical and Mathematical Correspondence*, ed. Brian McGuinness, trans. Hans Kaal [Chicago: University of Chicago Press, 1980], p. 182). In a letter to Hugo Dingler, dated 31 January 1917, Frege describes an inference from premises whose truth is not acknowledged as a "pseudo-inference" (*Philosophical and Mathematical Correspondence*, p. 17).

[16] See Wilfrid Sellars, "Naming and Saying," reprinted in *Science, Perception and Reality* (London: Routledge and Kegan Paul, 1963).

[17] "On the Scientific Justification," *Conceptual Notation*, p. 85.

[18] *Posthumous Writings*, p. 260.

[19] Whereas an object name or concept word "has its own specific meaning" (*Begriffsschrift* §1)—that is to say, it designates something, some particular object in the case of an object name and a certain concept in the case of a concept word—a Latin italic letter only "indicates an object, it does not have a meaning, it designates or means nothing" (*Posthumous Writings*, p. 190). An italic letter "simply does not have the purpose of designating a number, as does a number sign; or for that matter of designating anything at all" (*Collected Papers*, p. 307). "These letters . . . are not at all intended to designate numbers, concepts, relations, or some function or other; rather they are intended only to indicate so as to lend generality of content to the propositions in which they occur" (*Collected Papers*, p. 306). A Latin italic letter of *Begriffsschrift* does not designate

indeterminately or arbitrarily; for it does not designate at all. Nor does it have a sense: "a sign which only indicates neither designates anything nor has a sense" (*Posthumous Writings*, p. 249).

[20] We could, of course, dispense with the sign '&', using only signs for negation and the conditional. What is critical is the order of embedding of connectives, the fact that any of the connectives in '$S\supset(R\supset(Q\supset P))$' can be made the main connective given appropriate transformations.

[21] "On Mr. Peano's Conceptual Notation and My Own," in *Collected Papers*, p. 237.

[22] Frege tells us in *Begriffsschrift* §12 that a particular affirmative in his logical language can be read either as of the form 'some (at least one) S is P' or as of the form 'it is possible for an S to be a P'. If Frege's logical language is read quantificationally, the claim is manifestly false. As we read *Begriffsschrift*, it is true. The two English sentences correspond to two different analyses of Frege's two-dimensional presentation of a sense.

[23] "On the Aim of the 'Conceptual Notation'," in *Conceptual Notation*, p. 95.

[24] "Foundations of Geometry II," in *Collected Papers*, p. 338.

[25] "Thoughts," in *Collected Papers*, p. 352.