Several authors have argued that a version of Curry’s paradox involving validity motivates rejecting the structural rule of contraction. This paper criticizes two recently suggested alternative responses to “validity Curry.” There are three salient stages in a validity Curry derivation. Rejecting contraction blocks the first, while the alternative responses focus on the second and third. I show that a distinguishing feature of validity Curry, as contrasted with more familiar forms of Curry’s paradox, is that paradox arises already at the first stage. Accordingly, blocking the second or third stages won’t suffice for resolving the paradox.

Keywords  Curry’s paradox; validity; contraction-free logics; nontransitive logics

Introduction

Several authors, including myself, have argued that a version of Curry’s paradox involving a notion of validity motivates a radical revision to the system of rules used in deriving the paradox. The proposed revision involves rejecting the structural rule of contraction (Beall and Murzi, 2013; Mares and Paoli, Forthcoming; Priest, Forthcoming; Shapiro, 2011; Weber, Forthcoming; Zardini, 2013). Using a contraction-free proof system, what can be shown to follow from a collection of assumptions may depend on how many times a given assumption appears in that collection. This means that a contraction-free proof system can no longer treat collections of assumptions as sets.

Unsurprisingly, it has seemed to many that rejecting structural contraction is “very hard to get [one’s] head around” (Field, 2008, pp. 10–11), or at least that it stands in need of justification by a “new metaphysical account of validity” (Beall and Murzi, 2013, p. 164). It is therefore a pressing question whether there is a solution to “validity Curry” that preserves the standard conception of a collection of assumptions as a set. Two such solutions have recently been suggested: one is mentioned by Ketland (2012) and Bacon (Forthcoming), and a second is advocated by Ripley (2013).

The purpose of this paper is to argue that both solutions leave in place a strengthened version of validity Curry reasoning. There are three salient stages in a standard validity Curry derivation. Rejecting contraction blocks the first, while the alternative responses I will consider focus on the second and third. I show that a distinguishing feature of validity Curry, as contrasted with more familiar forms of Curry’s paradox, is that paradox can be obtained from the first stage, together with an uncontroversial principle.
concerning truth conditions. Accordingly, blocking the second or third stages won’t suffice for resolving the paradox.

**Validity principles**

In what follows, the “sequent” \( \Gamma \vdash \alpha \) should be understood as registering that sentence \( \alpha \) of the object language is a joint logical* consequence of the sentences in the set \( \Gamma \). (Sequents are expressions in our metalanguage; I allow context to disambiguate their use and mention.) By a “logical* consequence” of some assumptions, I mean anything that follows from them in virtue of (i) first-order logic, (ii) whatever principles of syntax or arithmetic are used to yield the self-reference appealed to below, and (iii) whatever inference rules govern the predicate Val introduced in the next paragraph. When \( \alpha \) is a logical* consequence of the sentences in \( \Gamma \), I will speak of the argument from \( \Gamma \) to \( \alpha \) as logically* valid. When \( \Gamma \) is empty, I will speak of \( \alpha \) as a logically* valid sentence.

Most importantly, I assume that the object language contains a binary predicate \( \text{Val} \) that expresses the relation of single-premise logical* validity: the sentence \( \text{Val}(\langle \alpha \rangle,\langle \beta \rangle) \) expresses the claim that \( \alpha \vdash \beta \). Just as the Liar paradox appears to show that a language can’t contain its own truth predicate, the validity Curry paradox appears to show that a language can’t contain its own logical* validity predicate. Each of the replies I will consider may be understood as an attempt to show that this conclusion is premature.

Our assumption that \( \text{Val}(\langle \alpha \rangle,\langle \beta \rangle) \) expresses the claim that \( \alpha \vdash \beta \) puts in place the following truth condition, expressed using a truth predicate in our metalanguage:

\[
(\text{VT}) \quad \text{Val}(\langle \alpha \rangle,\langle \beta \rangle) \text{ is true iff } \alpha \vdash \beta.
\]

As only the right-to-left direction of this biconditional will be relevant to my argument, let me give it a label:

\[
(\text{VT}_{\text{RL}}) \quad \text{If } \alpha \vdash \beta, \text{ then } \text{Val}(\langle \alpha \rangle,\langle \beta \rangle) \text{ is true.}
\]

However, presentations of the validity Curry paradox have instead invoked two principles concerning consequence, labeled Validity Detachment and Validity Proof by Beall and Murzi (2013):

\[
(\text{VD}) \quad \alpha, \text{Val}(\langle \alpha \rangle,\langle \beta \rangle) \vdash \beta.
\]

\[
(\text{VP}) \quad \text{If } \alpha \vdash \beta \text{ then } \vdash \text{Val}(\langle \alpha \rangle,\langle \beta \rangle).
\]

The standard version of validity Curry reasoning I will initially consider employs VD and VP together with the following version of the rule of Cut, which records the generalized transitivity of logical* validity:

\[
(\text{Cut}) \quad \text{If } \Gamma, \alpha \vdash \beta \text{ and } \Delta \vdash \alpha, \text{ then } \Gamma, \Delta \vdash \beta.
\]

In the following section, I lay out the standard validity Curry reasoning. Next, I explain why it has seemed that VP and Cut are open to challenge. However, I argue that there is a version of validity Curry reasoning that is just as paradoxical, but uses \( \text{VT}_{\text{RL}} \) in place of VP or Cut. Finally, I show how the availability of this strengthened version
distinguishes the validity Curry paradox from “ordinary” Curry paradox not involving a notion of validity.

**Validity Curry: three stages**

The validity Curry reasoning I will now present purports to establish, concerning any arbitrary sentence, that it is logically* valid. As my arbitrary sentence, I choose some absurdity ⊥, though for present purposes any claim that fails to be logically* valid would do equally well. (When we turn to the strengthened reasoning, however, it will matter that ⊥ is a sentence we wish to deny.) I also assume that in view of our theory of syntax or arithmetic the sentences κ and Val(⟨κ⟩, ⟨⊥⟩) are logically* equivalent.

It will be useful to divide the derivation of validity Curry paradox into three stages. In Stage 1, we reason that absurdity is a logical* consequence of the validity-Curry sentence:

\[
\begin{array}{c}
\vdots \\
\kappa, \text{Val}(⟨\kappa⟩, ⟨⊥⟩) \vdash ⊥ \\
\text{VD} \kappa \vdash \text{Val}(⟨\kappa⟩, ⟨⊥⟩) \\
\kappa \vdash ⊥ \\
\text{Cut}
\end{array}
\]

In Stage 2, using this subderivation, we establish that the validity-Curry sentence is logically* valid:

\[
\begin{array}{c}
\vdots \\
\text{Stage1} \kappa \vdash ⊥ \\
\text{Val}(⟨\kappa⟩, ⟨⊥⟩) \vdash \kappa, \text{Val}(⟨\kappa⟩, ⟨⊥⟩) \\
\text{VP} \kappa \text{Val}(⟨\kappa⟩, ⟨⊥⟩) \vdash ⊥ \\
\kappa \vdash ⊥ \\
\text{Cut}
\end{array}
\]

Finally, Stage 3 puts the previous two subderivations to use in establishing that absurdity is logically* valid:

\[
\begin{array}{c}
\vdots \\
\text{Stage1} \kappa \vdash ⊥ \\
\text{Stage2} \kappa \vdash ⊥ \\
\text{Cut}
\end{array}
\]

The challenge, of course, is to identify where this derivation goes wrong.

**Two replies that concede Stage 1**

According to Shapiro (2011), Beall and Murzi (2013), Zardini (2013) Mares and Paoli (Forthcoming), Priest (Forthcoming) and Weber (Forthcoming), the derivation should already be resisted at Stage 1. This is done by reformulating this subderivation in a “substructural” proof system that keeps track of how many times a given sentence appears in a collection of assumptions:

\[
\begin{array}{c}
\vdots \\
\kappa, \text{Val}(⟨\kappa⟩, ⟨⊥⟩) \vdash ⊥ \\
\text{VD} \kappa \vdash \text{Val}(⟨\kappa⟩, ⟨⊥⟩) \\
\kappa, \kappa \vdash ⊥ \\
\kappa \vdash ⊥ \\
\text{Cont}
\end{array}
\]

The subderivation can now be blocked by rejecting the final use of the structural contraction rule:
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(Cont) If $\Gamma, \alpha, \alpha \vdash \beta$, then $\Gamma, \alpha \vdash \beta$.

A second way to block Stage 1 would be to reject VD; one rationale for doing so might be Beall’s recent defence of a thoroughly “detachment-free” version of dialetheism (Beall, Forthcoming).

Here, I won’t examine the relative merits of these proposals for blocking Stage 1. Rather, my topic will be two replies that concede the conclusion of Stage 1. Thus Ketland (2012) focuses on the use of VP at Stage 2, and argues that VP is insufficiently motivated. Rejecting VP is also explored by Bacon (Forthcoming). Alternatively, Ripley (2013, p. 157) advocates blocking the derivation at Stage 3, by restricting the rule of Cut so as to invalidate this instance of its use. By stipulating that $\kappa$ is the very sentence $\text{Val}(\langle \kappa \rangle, \langle \bot \rangle)$, he does without Cut in the derivation’s first two stages, whose conclusions he endorses.

Consider first the status of VP. According to Priest (Forthcoming), VP “appears little more than definitional.” Presumably, he means that it follows from the meaning we have stipulated for the predicate Val. This claim seems mistaken. Given $\alpha \vdash \beta$, we should indeed be able to conclude, via VT$_{RL}$, that $\text{Val}(\langle \alpha \rangle, \langle \beta \rangle)$ is true. That much does immediately follow from the meaning of Val. But VP appears to go farther than this: it allows us to conclude that $\text{Val}(\langle \alpha \rangle, \langle \beta \rangle)$ is logically* valid. For this reason, as Ketland notes, VP is “problematic” (Ketland 2012, p. 428). Strictly speaking, Ketland rejects VP only on the assumption that Val is interpreted as expressing validity in first-order logic. However, he notes that paradoxes of validity might instead be taken to concern “some more general sense of ‘valid’” (2012, p. 427), and his criticism of the inference from VT$_{RL}$ to VP would seem to apply in that case as well. Bacon, too, calls attention to the “somewhat controversial” status of VP (Forthcoming, §2.3).

As for Cut, Ripley questions the rule’s intuitive obviousness by adopting an interpretation of validity from Restall (2005). On his interpretation of the turnstile employed in stating the nontransitive logic he proposes in response to semantic paradox, the sequent $\kappa \vdash \bot$ tells us that it is “out of bounds” to assert $\kappa$ while denying $\bot$. Likewise, the sequent $\bot \vdash \kappa$ tells us that it is out of bounds to deny $\kappa$. But from the fact that $\kappa$ is undeniable, it doesn’t follow that we must assert it. Hence, the sequents $\kappa \vdash \bot$ and $\bot \vdash \kappa$ don’t jointly preclude us from denying the absurdity $\bot$. As Ripley interprets the turnstile, then, they don’t establish $\bot$.

In short, we have two responses to validity Curry which concede that the validity-Curry sentence $\kappa$ carries $\bot$ as its logical* consequence. The first response, suggested by Ketland and Bacon, denies that it follows that $\kappa$ is logically* valid. The second response, endorsed by Ripley, concedes that $\kappa$ is logically* valid, but denies that there is anything problematic about that conclusion.

Why Stage 1 already yields paradox

This paper won’t examine the status of VP or Cut. Instead, I now argue that neither rejecting the VP step of Stage 2 nor rejecting the Cut step of Stage 3 will resolve the validity Curry paradox.
To see why, first observe a peculiarity of Ripley’s proposal that we concede Stages 1 and 2. Suppose, as is natural, that a logically* valid sentence must at least be true. In that case, the conclusions of Stages 1 and 2, namely the sequents $\kappa \vdash \bot$ and $\vdash \kappa$, would tell us that an absurdity follows logically* from $\kappa$ and also that $\kappa$ is true. Yet the appearance that an absurdity follows from claims one takes to be true is surely the essence of paradox.

Thus, it is crucial to Ripley’s response to validity Curry that he not accept the claim that logically* valid sentences are true. Indeed, as we have seen, he construes them instead as undeniable. So far, then, I have raised no objection: his concession that $\vdash \kappa$ doesn’t commit him to the problematic concession that $\kappa$ is true. But the lesson of these reflections is more general. No solution to validity Curry can be adequate if it concedes both of the following:

(a) $\kappa \vdash \bot$
(b) $\kappa$ is true.

It should now be apparent what is wrong with both proposals for blocking validity Curry while conceding Stage 1. Once we concede that $\kappa \vdash \bot$, $\text{VT}_{RL}$ tells us we have conceded that $\text{Val}(\langle \kappa \rangle, \langle \bot \rangle)$ is true. Furthermore, this suffices to show that $\kappa$ is true as well, given the principle

$$
\text{(TE)} \text{ If } \alpha \text{ is true, and } \alpha \text{ is logically* equivalent to } \beta \text{, then } \beta \text{ is true.}
$$

In short, in the presence of both VT and TE, condition (a) above seems to imply condition (b), and we have seen that, taken together, they are paradoxical.

But VT merely spells out a consequence of our assumption that Val expresses single-premise logical* consequence, and TE is an extremely natural principle. Bacon counts TE as “part of the naive conception of truth.” And there is no reason why Ripley’s conception of validity should preclude him from accepting TE. In any case, though, his own stipulation that $\kappa$ is the very sentence $\text{Val}(\langle \kappa \rangle, \langle \bot \rangle)$ makes appeal to TE unnecessary.

I conclude that given the intended interpretation of the predicate Val, Stage 1 of our derivation suffices for paradox. If this is right, our options for resisting validity Curry paradox are more limited than Ketland, Bacon or Ripley suggest. There is a version of validity Curry reasoning that can’t be evaded by rejecting the instance of VP questioned by Ketland and Bacon or the instance of Cut rejected by Ripley.

**Ordinary Curry paradox**

In this regard validity Curry differs from usual versions of Curry’s paradox, where the strengthened reasoning is unavailable. What Beall and Murzi (2013) call “the standard conditional-Curry paradox” is formulated using some detaching conditional $\to$, together with a truth predicate. Here it is essential that $\to$ isn’t understood as tracking logical* consequence. In other words, it is not assumed that $\alpha \to \beta$ is true iff $\alpha \vdash \beta$.

For easier comparison with validity Curry, I will formulate an ordinary Curry paradox using a binary predicate $\text{Imp}(x,y)$ in place of $\to$. To do so, assume that the logical* consequence expressed by the turnstile also encompasses an equivalence between $\text{Imp}(\langle \alpha \rangle, \langle \beta \rangle)$ and $\alpha \to \beta$. Since $\to$ detaches, we thus have
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(ID) $\alpha, \text{Imp}(\langle \alpha \rangle, \langle \beta \rangle) \vdash \beta$.

But the important fact is that we are not assuming

(IT) $\text{Imp}(\langle \alpha \rangle, \langle \beta \rangle)$ is true iff $\alpha \vdash \beta$.

Suppose, now, that there is a sentence $\gamma$ that is logically* equivalent to $\text{Imp}(\langle \gamma \rangle, \langle \bot \rangle)$. Using ID in place of VD, Stage 1 lets us derive the sequent $\gamma \vdash \bot$. This time, however, we can block the easy inference to the claim that $\gamma$ is true. To do so, we must reject

(ITR$_L$) If $\alpha \vdash \beta$, then $\text{Imp}(\langle \alpha \rangle, \langle \beta \rangle)$ is true.

Whereas rejecting VTR$_L$ is incompatible with taking Val to express logical* validity, the intended interpretation of Imp won’t immediately rule out rejecting ITR$_L$. This difference explains why Stage 1 of our validity Curry derivation already counts as paradoxical, whereas the corresponding stage of an ordinary Curry derivation does not.

If this is right, the difference between validity Curry and ordinary Curry has been mischaracterized. According to Beall and Murzi (2013, p. 156), the key contrast is that ordinary Curry can be resolved by joining Field (2008) and Beall (2009) in giving up the “deduction-theorem link”

(IP) If $\alpha \vdash \beta$, then $\vdash \text{Imp}(\langle \alpha \rangle, \langle \beta \rangle)$,

whereas “giving up VP seems not to be an option” for resolving validity Curry. I have elsewhere drawn the same contrast (Shapiro, 2011, pp. 336–37). But even if rejecting VP turns out not to be an option, that doesn’t explain the resilience of validity Curry. For we have seen that validity Curry does not in fact require VP. Instead, the important contrast is that ordinary Curry can be resolved by giving up ITR$_L$, whereas giving up VTR$_L$ isn’t an option.

Conclusion

I have criticized two responses to validity Curry paradox that preserve structural contraction. The problem is that both responses concede Stage 1 of the validity Curry derivation, and this leads to paradox in the presence of VT, a principle which falls directly out of the intended interpretation of the predicate Val. To avoid paradox, it seems, we must instead find a way to block Stage 1. As long as we continue to maintain principle VD, according to which the validity relation detaches, this conclusion speaks in favor of a contraction-free reply to validity Curry.

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Notes

1 An earlier discussion of the paradox is found in Whittle (2004); for the history of related paradoxes of validity, see especially Read (1979) and the helpful overview in Beall and Murzi (2013, pp. 153–56).

2 Zardini (2011) and Mares and Paoli (Forthcoming) suggest different metaphysical justifications. In Shapiro (Forthcoming), I argue that contraction-free proof systems require no metaphysical justification.

3 Formulating the paradox in terms of logical* validity allows me to circumvent difficulties raised by Cook (Forthcoming) and Ketland (2012) for the claim that there is a Curry paradox involving “logical validity.” These difficulties rest on the extra-logical status of (ii) and arguably (iii). Cook argues that the status of (iii) is the real obstacle to a paradox of logical validity. For discussion, see also Murzi and Shapiro (Forthcoming, §2.3).

4 Accordingly, one response is to embrace instead a hierarchy of validity relations such that Val_j expresses the notion of an argument’s being valid in virtue of the principles governing Val_k for j < k, together with logic and the resources for self-reference. See Whittle (2004) and Myhill (1975).

5 In Shapiro (Forthcoming), I defend an interpretation of sequent structure on which contraction-free and detachment-free responses to validity Curry need not be in conflict, once they are taken to involve different understandings of the structural comma.

6 See Ripley (2012, p. 355) for an explanation of this stipulation. Here ⟨α⟩ serves as his notation for whatever individual constant is the language’s “distinguished name” for α. An alternative way of formulating Stages 1 and 2 without Cut would use the other means of securing self-reference Ripley mentions (citing Kremer 1988). Here the angle brackets are the object language’s own device for forming quote names. Let κ be the sentence Val⟨c,⟨⊥⟩⟩, and interpret a sequent α ⊢ β as saying that β follows from α together with the identity c = ⟨κ⟩, by first-order logic (with identity) and the logic of Val. Starting once again with κ, Val⟨⟨κ⟩,⟨⊥⟩⟩ ⊢ ⊥, we now get κ ⊢ ⊥ and ⊢ κ without using Cut. (Thanks to Dave Ripley for discussion that led to this note.)

7 He points out that Beall and Murzi (2013, p. 152) identify VT as “the basic idea” behind VP.

8 However, as a referee pointed out, Ketland might also reject VT, by denying that a language can contain its own logical* validity predicate. His paper’s purpose is to show that Peano Arithmetic can be extended conservatively to include a predicate that expresses classical first-order validity.

9 This is also apparent from Ripley’s response to the Liar paradox, which concedes that the liar sentence is logically* valid, even though it has absurdity as a logical* consequence, whence it is out of bounds to assert the liar sentence while denying absurdity.

10 In concluding, by reflection on the strengthened validity Curry reasoning, that no adequate solution to validity Curry can concede (a), I have in effect reasoned as follows: since (a) entails (b), and (a) and (b) taken together entail ‘⊥ is true’, then (a) by itself entails that sentence. This inference presupposes structural contraction. However, that is unproblematic in the present context, where I am challenging proposed solutions that retain contraction. (Thanks to Dave Ripley for pointing this out.)

11 Ripley does show that his proposed logic endorses TE as logically* valid when TE is formulated in the object language using a validity predicate governed by VD (2013, p. 158). As we have seen, however, he is not prepared to assert all sentences he regards as logically* valid.
Nor is TE required if we use the alternative means of securing self-reference sketched in note 6. In that case, the conclusion by VTRL that Val(⟨κ⟩, ⟨⊥⟩) is true tells us—given the stipulation that c names κ—that Val(c, ⟨⊥⟩), namely κ, is true as well.

Citing Whittle (2004) and Shapiro (2011), Beall and Murzi note that validity Curry can also be formulated using a truth predicate and a connective ⇒ such that α ⇒ β is true iff α ⊢ β.

Beall and Murzi focus on the connective formulation of ordinary Curry, but they note that the contrast applies to the predicate formulation as well (2013, p. 154). In Shapiro (2011), I focus on connective formulations of both ordinary and validity Curry, and state the contrast between them thus: while we might reject Conditional Proof for →, we can’t reject Conditional Proof for an “entailment connective” ⇒ such that α ⇒ β is equivalent to Val(⟨α⟩, ⟨β⟩).

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