

Mathematical Infinity and the Presocratic Apeiron

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Abstract: The Presocratic notion of *apeiron*, often translated as “unbounded,” has been the subject of interest in classical philosophy. Despite apparent similarities between *apeiron* and infinity, classicists have typically been reluctant to equate the two, citing the mathematically precise nature of infinity. This paper aims to demonstrate that the properties that Anaximander, Zeno, and Anaxagoras attach to *apeiron* are not fundamentally different from the characteristics that constitute mathematical infinity. Because the sufficient explanatory mathematical tools had not yet been developed, however, their quantitative reasoning remains implicit. Consequentially, the relationship between infinity and *apeiron* is much closer than classical scholarship commonly suggests.

The ἄπειρον¹ or *apeiron*, a recurring theme in the history of Greek philosophy, is first mentioned in fragments of Anaximander, whose abstract characterization of the word has been the source of some contention in Presocratic scholarship. Ostensibly, the word is taken to mean “unbounded,” “unlimited,” or “unfinished,” and, in accordance with the variety of translations, the word is put to a variety of uses within the interpretation of its function in Presocratic philosophy. In its earliest observable form, the word appears in context of cosmogony, but it is clear that since its historical origin, the Unbounded has played many philosophic roles—as a divine progenitor, fundamental substance, or quantitative entity, to name a few—for many different philosophers in the progression of Greek thought. As such, this paper will focus on an examination of several

¹ *Perseus Digital Library*, s.v. “ἄπειρον,” ed. Gregory R. Crane, Tufts University, accessed January 15, 2014, <http://www.perseus.tufts.edu/hopper/morph?l=a%29/peiros&la=greek&prior=pe/ras&d=Perseus:text:1999.04.0057:entry=a%29pei/rwn2&i=1#lexicon>.



different uses of the Unbounded in Presocratic philosophy, albeit through the unusual lens of the modern mathematical infinite, as described within set theory.

In order to understand any of the similarities or differences the two concepts might hold, a brief sketch of the current notion of infinity will first be necessary. Currently, the exploration of infinity is now largely contained within the field of set theory, which in turn studies the properties of collections comprised of objects in an encompassing organizational entity, called a set. For example, the set $\{2, 4, 6\}$ contains 3 distinct elements: 2, 4, and 6. In this mathematical structure, it is possible to exhibit a variety of sets with interesting mathematical concepts. Most relevant to the discussion at hand are sets which seem to contain an inexhaustible number of entities, such as the set of all positive integers $\{1, 2, 3, \dots\}$, or the set of all prime numbers $\{2, 3, 5, \dots\}$. It is when we examine sets such as these that a picture of infinity begins to emerge. If we were asked to determine which of those two sets (the set of all positive integers and the set of all positive prime numbers) is larger, the answer would not be obvious. On the one hand, the former set necessarily contains all the members of the latter set, as well as divisible numbers not contained in the latter set. But on the other, if we were to line up one entity from the set of positive integers with one entity from the set of all positive even numbers in such a way that the first entity in both sets were paired, then the second entity in both sets, then the third, then the fourth, and so on, we would find that both sets appear to have an equal number of members. This process of bijection or one-to-one correspondence, put to use most notably in Galileo's *Two New Sciences*, demonstrates the odd properties of sets that have a seemingly endless number of members and was ultimately developed into the definition of infinity used today.² If a set can be placed into this one-to-one correspondence with one of its proper subsets (meaning simply that all the members of the subset are contained within the superset, as well as other entities not found in the subset), then that set is said to contain an infinite number of members.³ This definition, first laid out in the twentieth century by the

² Galileo Galilei, *Two New Sciences* (Madison: University of Wisconsin Press, 1974), 39-42.

³ Richard Courant and Herbert Robbins, *What Is Mathematics? An Elementary Approach to Ideas and Methods* (New York: Oxford University Press, 1941), 77-81.

German mathematician Georg Cantor, would serve as the basis from which infinity could be studied as a mathematical concept, marking a paradigmatic change in the way discourse about the infinite would take place.⁴ The inception of this discourse, however, is owed to the Greek *apeiron*, and we shall spend the remainder of our time examining both its characterizations and their similarities to the modern infinite.

Despite the multitude of descriptions of *apeiron* from philosophers both ancient and contemporary, there has been hesitation from both philosophers and historians when attempting to critically analyze the concept in conjunction with its modernized equivalent. Philip Wheelwright cautiously warns the reader that "the most nearly accurate translation would be 'the Qualitatively Unlimited,'" shying away from any quantitative associations because "the word 'infinite' has technical associations . . . which may render it misleading for so early a mode of thought."⁵ Wheelwright's intention may seem, on its face, like a simple clarificatory remark, but he has nonetheless drawn a firm distinction between the qualitative *apeiron* and the quantitative infinite. Wheelwright is not alone in drawing attention to this distinction. James Wilbur goes so far as to state, "It is generally agreed upon that to call it [*apeiron*] 'infinite' . . . is a mistake," since "the idea of the infinite with its mathematical implications is much too complicated to be used here."⁶ The concerns are well-founded. While the morphological similarities between the two words might seem to suggest an obvious equivalence (both derive from the negation of the root word, "finite" in English and "*peirar*" or "limit" in Ancient Greek),⁷ there are certainly reasons to hesitate before offering a direct comparison. As we have seen, the word "infinite" has taken on a precise mathematical definition and, as such, has gradually ceased to be a topic of solely philosophical investigation. Interestingly, almost the opposite story can be seen emerging from the Greek picture of *apeiron*. From its relatively clear origin as a divine force of creation, it gradually became a trait synonymous with the

⁴ Ibid.

⁵ Philip Wheelwright, *The Presocratics* (New York: Odyssey Press Inc., 1966), 53-54.

⁶ James Wilbur, *The Worlds of the Early Greek Philosophers* (Buffalo, NY: Prometheus Books, 1979), 37-38.

⁷ *Perseus Digital Library*, s.v. "πεῖραρ," ed. Gregory R. Crane, Tufts University, accessed January 15, 2014, <http://www.perseus.tufts.edu/hopper/text?doc=Perseus%3Atext%3A1999.04.0057%3Aentry%3Dpei%3Drar>.



indeterminate, undefined, or imperfect, losing the explicit function it previously served. However, it may be the case that upon a close and careful reading of some of the texts discussing the Unbounded, some quantitative comparisons between the *apeiron* and the infinite may be drawn.

In the first accounts of the *apeiron* by Anaximander, it is clearly represented as a divine figure, transcending the material world through its unbounded nature in space or time.⁸ In essence, these two traits, divinity and temporal endlessness, were synonymous. Diogenes Laërtius ascribes a number of sayings to Thales of Miletus, among them an allusion to the eternal nature of the divine: “What is divine? That which has neither beginning nor end.”⁹ As a student of Thales, Anaximander himself likely had a similar picture of the divine and represents the *apeiron* as the original entity which creates and guides the world. The process of creation achieved through the *apeiron* varies according to the commentator, however. Aristotle and Aëtius describe a process whereby the form is spun out of the formless, establishing elemental opposites such as hot and cold which then combine in different concentrations to form material objects.^{10,11} Later, some philologists and philosophers have interpreted Anaximander’s Unbounded as a formless, endless mass, out of which the material objects emerge, only to gradually return to the shapeless whole,¹² while others have identified it as the vessel in which the material world or worlds reside: an ever-present, temporally unbounded background.¹³ Still others have argued that the *apeiron* was intended to be the endless cyclical process of creation and destruction itself.¹⁴ Regardless of

⁸ Theo Gerard Sinnige, *Matter and Infinity in the Presocratic Schools and Plato* (Assen: Koninklijke Van Gorcum & Company, 1968), 5-7.

⁹ Diogenes Laërtius, *Lives of Eminent Philosophers*, ed. Tiziano Dorandi (Cambridge: Harvard University Press, 1980), 37.

¹⁰ Aristotle. *Phys. I.4*, 187a20, trans. Hardie and Gayle.

¹¹ Arthur Fairbanks, *The First Philosophers of Greece* (London: K. Paul, Trench, Trubner, 1898), 15.

¹² John Burnet, *Greek Philosophy* (London: Macmillan and Company, 1914), 23.

¹³ W.K.C Guthrie, *The Greek Philosophers: From Thales to Aristotle* (Abingdon: Methuen & Co. 2012), 25-26.

¹⁴ Elizabeth Asmis, “What is Anaximander’s *Apeiron*?” *Journal of the History of Philosophy* 19 (1981): 279-297.

the specific nature of Anaximander's *apeiron*, however, there is one particular characteristic common to all analyses which will prove to be a unifying aspect of the many different conceptions of the *apeiron*. In essence, the use of the term "*apeiron*" always consists of more than negation of the finite, which is generated from it. It is the relationship between the Bounded and the Boundless which will prove to be the most dynamic and variable aspect of Presocratic theories about the *apeiron*. Aëtius claims in *Placita Philosophorum* that Anaximander posited the *apeiron* as a generative force that is necessarily unending, claiming, "For what other reason is there of an Infinite but this, that there may be nothing deficient as to the generation or subsistence of what is in Nature?"¹⁵ Aristotle himself gives the very similar reasoning in Book 3 of the *Physics* in explaining the metaphysical appeal of the *apeiron* for past philosophers.¹⁶ These arguments entail a contrast between the finite and the infinite, which, when understood in conjunction with the unbounded principles by which the *apeiron* generates the finite, appears mathematical in nature.

If we temporarily assume that Anaximander intended his cosmic system of separation and re-amalgamation to entail an endless number of co-existing, spatially finite worlds in the embrace of the Boundless, the argument of the necessity of the Unlimited is grounded in quantitative reasoning. It implies that Anaximander understood that an endless number of temporary worlds, regardless of size, could only be generated from a similarly endless quantity of matter. Mathematically represented, this is surprisingly close to the definition of infinity in modern set theory. Taking each world-order as an entity in the endless collection of world-orders, Anaximander is claiming that the set of world-orders is a subset of the entities which are generatable by the *apeiron*. Provided we accept the premise that matter is conserved between objects and their generative source, it is only a small intellectual jump (albeit one not made explicitly by Anaximander or Aristotle) to place the set of world-orders and the set of entities generatable by the *apeiron* in one-to-one correspondence, demonstrating the quantitatively infinite nature of Anaximander's *apeiron*.

While it is certainly far-fetched to claim Anaximander had an intuitive understanding of set theory, it is not so unbelievable that he could recognize some of the quantitative characteristics inherent to his

¹⁵ Diogenes Laërtius, *Lives of Eminent Philosophers*, 37.

¹⁶ Aristotle. *Phys.* III.4, 203b7, trans. Hardie and Gayle.



apeiron without possessing the vocabulary to explicate its mathematical nature. Nor is this interpretation limited to the spatially-coexistent interpretation of Anaximander's cosmology. If anything, it is more apparent in the case where we compare the infinite series of finite cycles of generation and destruction undergone by a single material world with the eternal nature of *apeiron*. In describing an infinite number of temporary generative cycles, only an immutable *apeiron*, undergoing no changes, could serve as an equivalent source which will never fail to exist throughout time. Rather than spatial or material sources, we can speak in terms of an endless set of world-cycles and its temporally eternal source.¹⁷

Over time, the view of *apeiron* as a creative and destructive divine figure gave way to the Pythagorean view of a central dichotomy in which the bounded and the boundless were set in opposition. The material world, composed of limits and boundaries, continually suppresses and binds the unlimited into physical reality. In the Pythagorean view, the world is composed of finite things, which can be rationally understood through mathematics, set in warring opposition with the Unlimited, which cannot be understood or examined.¹⁸ Accordingly, the nature and properties of *apeiron* became its lack of definition and apparent irrationality, properties to be avoided by the rationally minded Pythagoreans and their successors. As we shall see, however, the conversation about the quantitative nature of the Boundless did not end with the Pythagoreans but can be seen in fundamental mathematical problems highlighted by Zeno's paradoxes.

Typically interpreted as a defense of Parmenidean monism, Zeno's paradoxes are a variety of *reductio ad absurdum* arguments targeted primarily at revealing the untenable consequences of a discontinuous reality and the motion of objects. Correspondingly, there are only "paradoxes" insofar as they appear to contradict obvious empirical evidence—for Zeno, they are arguments for the existence of Parmenides's Being.¹⁹ While all four of the paradoxes (as outlined in

¹⁷ It should be noted, however, that this argument does not work with regards to Elizabeth Asmis's interpretation of Anaximander's *apeiron* as equivalent to the very cyclical process the world undergoes, since no contrast between the cycle and a second entity is ever established.

¹⁸ Aristotle. *Met. I.5*, 987a9-27, trans. Ross.

¹⁹ Nick Huggett, "Zeno's Paradoxes," *The Stanford Encyclopedia of Philosophy* (Winter 2010), accessed March 12, 2014, <http://plato.stanford.edu/archives/win2010/entries/paradox-zeno/>.

Aristotle's *Physics*) vary in their potency and coherence, the classicist Theo Sinnige identifies two presuppositions on which the *reductio* arguments are conducted: "(1) that reality is discontinuous, (2) that there is no limit to this discontinuity, i.e. that the theoretically infinite divisibility of a mathematical magnitude is also applicable to spatial magnitudes."²⁰ It is from these two suppositions that the rest of the paradoxes (at least, those concerning spatial reasoning) are built.

The simplest construction is seen in the Stadium, or Dichotomy argument, in which an athlete begins running from a specific point, p_0 , in hopes of reaching the finish line at point P_1 .²¹ Before reaching p_1 , however, the runner must first pass $P_{1/2}$, a point stationed between the starting line and the finish line, and then $P_{3/4}$, then $P_{7/8}$ and so on, until it is clear that he must pass through an infinite number of closer and closer points before reaching P_1 . It is not possible, Zeno concludes, to pass through an infinite number of points in a finite period of time, and so the runner will never reach the finish line, or move at all for that matter; regardless of how small the space is between p_0 and p_1 , there will always be an infinite number of intermediary points which are impossible to cross in a finite span of time.

In order to appreciate the significance of the Dichotomy paradox in regards to a geometric or mathematical notion of infinity, it is important to keep in mind the original definition of *apeiron* was that of an entity without limits. The spatial paradoxes of Zeno are not simply mathematical representations of a variety of infinitely divisible processes. They also form an implicit criticism of the simple view of *apeiron* as any process repeated without end. In these paradoxes, Zeno is concerned with the cardinal number of points within any line, which he properly identifies as being limitless via division. Zeno is presenting the existence of the boundless number of points within any description of bounded space, a notion which defied the traditional irreconcilable dichotomy of *apeiron* and *peiras*. By bringing the two features of Pythagorean philosophy in conflict, Zeno is pointing out the limitations implicit in the previously-held description of the *apeiron* as simply a thing without bounds. In this sense, the paradoxes are a challenge to either abandon the notion of mathematical, discontinuous

²⁰ Theo Gerard Sinnige, *Matter and Infinity in the Presocratic Schools and Plato* (Assen: Koninklijke Van Gorcum & Company, 1968), 89.

²¹ Aristotle, *Physics*, trans. R. Hardie and R. Gayle, *Internet Classics Archive*, <http://classics.mit.edu/Aristotle/physics.html>.



space or to revise the notion of *apeiron* to something that can be used and manipulated mathematically. Both Anaxagoras and Aristotle would take the latter route, although they would arrive at very different notions of a mathematical revision of *apeiron*.

As a contemporary of Zeno, Anaxagoras was most likely familiar with the paradoxes as well as the works of the Eleatic School preceding him. Nonetheless, his portrayal of the origins of the world is noticeably different from either Anaximander or a Parmenidian monism, best summarized in a brief fragment recorded by Simplicius: “All things were together; infinite both in number and in smallness; for the small also was infinite [ἄπειρον]. And when they were all together, nothing was clear and distinct because of their smallness.”²² Anaxagoras immediately sets a chaotic picture of this primordial entity as consisting of infinitesimal parts which then undergo a process of homogenizing or “separating out,” not unlike the process undergone by *apeiron* in the theories of Anaximander. Unlike Anaximander, however, Anaxagoras has included a curious statement identifying *apeiron* with “smallness,” which, as we will see, retains and demonstrates an understanding of the abstract complexity of the infinitely divisible.

In the fragments of Anaxagoras, an understanding is present of the concept of a group possessing some number of elements within it, an idea that would later develop into the mathematical set. As we have already seen, Zeno’s paradoxes establish the idea of infinite multiplicities contained in finite lengths. While this idea of a multiplicity may hint at the future development of a more rigorous conception of mathematical sets, Zeno stops short of examining the concept of a multiplicity itself and the quantitative properties it holds. Anaxagoras, however, takes up this challenge. In the fifth remaining fragment, he writes, “The sum total of all things is not a bit smaller nor greater; for it is not practicable that there should be more than all, but the sum total is always equal to itself.”²³ Like many other preceding notions about sets and multiplicities, the mathematical role that this fragment plays in the reasoning of Anaxagoras about infinitely divisible multiplicities must be teased out. The “sum total of all things” in regards to the Achilles paradox are undeniably finite and yet contain a notion of infinite divisibility that could, in a more mathematical setting, be understood

²² Arthur Fairbanks, *The First Philosophers of Greece* (London: K. Paul, Trench, Trubner, 1898), 238.

²³ Sinnige, *Matter and Infinity*, 129.

as the infinite set of rational numbers contained between the numbers 0 and 1 on a number line. Zeno's original paradoxes questioned the possibility of Achilles to overtake the tortoise through the infinite division of a finite length by assuming the necessary incompleteness of reasoning with infinite series, but Anaxagoras demonstrates in the fifth fragment that the totality of all things (in this case, an infinite series), can in fact be taken as a whole and completed quantity or, as would be later developed by Bolzano and Cantor, as a completed infinite set.^{24, 25} The finished picture is a complete mathematical revision of the concept of *apeiron* into a concept resembling the modern notion of the infinite. In addition to the older sense of a quantitative *apeiron* without an upper limit, there is now a notion of a completed infinite multiplicity of parts, which can be referred to and manipulated as a mathematical entity. The two senses of *apeiron* are combined in another fragment of Anaxagoras in which he talks specifically about parts in a whole: "There are just as many parts in the great as in the small taken as a multitude."²⁶ Mathematically, then, Anaxagoras has placed the two quantitative uses of *apeiron* (entities in an infinitely divisible length and entities within an infinite magnitude) in one-to-one correspondence with each other: another step towards a mathematically rigorous definition of the infinite.

Despite their ingenuity and subtlety, Anaxagoras's perspective on *apeiron* and the quantitative problems of Greek philosophy was not developed beyond the philosopher's original thought, owing to a variety of potential factors. While the fragments of Anaxagoras have substantial mathematical implications, it is clear that their intended purpose was to describe a naturalist cosmogony. The ultimate importance of the principles of mathematical divisibility and notions of equality between the large and the small were to explain how physical objects could aggregate from elemental chaos and still contain minuscule portions of all other things. This in turn was made to support theories regarding how many natural objects (bodies and plants, for example) grow over time. As such, the mathematical reasoning used to support Anaxagoras's physical theories was not the focus of his own inquiry.

²⁴ Fairbanks, *The First Philosophers of Greece*, 237.

²⁵ Aristotle. *Met.* IX.6, 1048b1-20, trans. Ross.

²⁶ *Ibid.*, 129.



The *apeiron* certainly deserves a spot on the genealogy of infinity, but its place is unclear. Its ambiguous origins and fluid definition compound the problem of interpreting its use in a variety of philosophical contexts. However, its watermark can be seen in a variety of Presocratic theories, even in cases where the theorists themselves avoid its explicit use. Whenever the *apeiron* is given an ontological role to play, when the endless is made physical, these thinkers had no choice but to confront the quantitative implications of such an entity and wrestle with the same problems which would later engage mathematicians (albeit in a more semantically precise field). To claim that Anaximander preempted Galileo's bijection or that Bolzano's sets were first developed by Anaxagoras would be an overstatement of the evidence at hand, but the manner in which these Presocratic thinkers handle the *apeiron* suggests a struggle to grasp the peculiar mathematical characteristics of infinity without access to a mathematical structure that would arrive more than a thousand years later.^{27, 28}

²⁷ Edgar, Morscher, "Bernard Bolzano -11.2: Preparatory Writings in Set Theory", *The Stanford Encyclopedia of Philosophy* (Spring 2013 Edition), Edward N. Zalta (ed.), accessed March 19, 2014, <http://plato.stanford.edu/archives/spr2013/entries/bolzano/>.

²⁸ José, Ferreirós, "The Early Development of Set Theory", *The Stanford Encyclopedia of Philosophy* (Winter 2012 Edition), Edward N. Zalta (ed.), accessed March 19, 2014, <http://plato.stanford.edu/archives/win2012/entries/settheory-early/>.