

FREGE'S WAYS OUT

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Abstract:

I show that Frege's statement (in the Epilogue to his Grundgesetze der Arithmetik, v. II) of a way to avoid Russell's paradox is defective, in that he presents two different methods as if they were one. One of these "ways out" is notably more plausible than the other, and is almost surely what Frege really intended. The well-known arguments of Lesniewski, Geach, and Quine that Frege's revision of his system is inadequate to avoid paradox are not affected by the ambiguity of Frege's statement. But a recent argument by Linsky and Schumm (Analysis 32 (1971-72), 5-7), intended as a very simple derivation of a contradiction within Frege's revised system, is valid only for the less plausible of the two versions of Frege's way out, and thus is not an effective attack on the revision that Frege intended to make.

## Frege's Ways Out

Discussions of "Frege's Way Out" commonly fail to mention that Frege had two different ways out which are of unequal merit. I shall document their existence and then compare them.

The relevant text is the Epilogue to the Grundgesetze der Arithmetik, v. II, in which Frege reports Russell's Paradox and revises his own system (in particular, Basic Law (V)) in an attempt to avoid it. After some discussion of the paradox he introduces his revision in the following words (as translated by M. Furth, The Basic Laws of Arithmetic, p. 139):

We see that the exceptional case is constituted by the extension itself, in that it falls under only one of two concepts whose extension it is. . . . Accordingly the following suggests itself as [the] criterion of identity for extensions: the extension of the one concept coincides with that of another if every object that falls under the first concept, except the extension of the first concept, also falls under the extension of the second concept, and if conversely every object that falls under the second concept, except the extension of the second concept, also falls under the first concept. (Gg., v. II, p. 262.)

This is expressed with admirable clarity. The most direct translation into a formula yields:

$$(1) \hat{X}Fx = \hat{X}Gx \cdot \equiv (w) (w \neq \hat{X}Fx \cdot Fw \cdot \supset Gw : w \neq \hat{X}Gx \cdot Gw \cdot \supset Fw).$$

(I have translated 'if' by ' $\equiv$ ' because Frege calls this 'the criterion' of identity, and also because of what follows.)

A few lines further down the page Frege writes: "By transferring to courses-of-values in general what we have said of extensions of concepts, we arrive at the Basic Law (V') which is to replace (V)." The formula which he produces here does indeed apply to courses-of-values in general; but for present purposes we may confine our attention to extensions of concepts. The formula is automatically so restricted when it is converted into modern notation, resulting in:

$$(2) \hat{X}Fx = \hat{X}Gx \cdot \equiv (w) (w \neq \hat{X}Fx \cdot w \neq \hat{X}Gx : \supset \cdot Fw \equiv Gw).$$

Clearly Frege regards (1) and (2) as equivalent; but clearly he is wrong. For according to (2), in determining whether the extensions of

two concepts are the same we determine whether the concepts apply to just the same things, leaving out of consideration the extensions themselves. But according to (1) we are to proceed by examining the things that fall under the first concept, excepting the extension of the first concept, but not necessarily excepting the extension of the second concept, to see if these all fall under the second concept too; and to examine the things that fall under the second concept, excepting the extension of the second concept, but not necessarily excepting the extension of the first concept, to see if they all fall under the first concept. It is intuitively obvious that we need not get the same result from these two different operations, and in fact that (2) implies (1) but not vice-versa.

An illustration is provided by the empty set  $\wedge$  (defined as  $\hat{\wedge}(x \neq x)$ ) and its singleton  $\iota\wedge$  (defined as  $\hat{\iota\wedge}(x = \wedge)$ ); that is, by substituting  $\xi \neq \xi$  for  $F\xi$  and  $\xi = \wedge$  for  $G\xi$  in the above formulas. With (2) as a premise it is quite easy to prove ' $\wedge = \iota\wedge$ .' For neither of the two concepts applies to anything aside from  $\wedge$  itself; and  $\wedge$  may be dismissed from consideration since it is the extension of one of the concepts. On the other hand, from (1) we cannot in a similarly easy and direct fashion derive the same conclusion. To be sure, everything even without exception that falls under  $\xi \neq \xi$  (*viz.*, nothing) falls under  $\xi = \wedge$ . But under  $\xi = \wedge$  there falls  $\wedge$ , which certainly does not fall under  $\xi \neq \xi$ ; and whether  $\wedge$  is the extension of  $\xi = \wedge$  is the very point at issue. The fact that  $\wedge$  is the extension of the first concept  $\xi \neq \xi$  is here irrelevant.

What would Frege have thought if he had noticed the discrepancy between (1) and (2)? Both textual and logico-philosophical evidence indicate that he would have taken (1) as his new version of Basic Law (V) and discarded (2).

In the text Frege presents (1) first, and in prose. Only then does he present (2), and he offers it as a symbolic formulation of (1). It is thus likely that he made a slip in transferring his thought from ordinary language to logical symbolism, and that (1) expresses his real intentions.

Furthermore (2) has some unwelcome immediate consequences which do not so obviously follow from (1). Consider, for example, ' $\wedge = \iota\wedge$ .' Frege might have been expected to look askance at this, for it is an instance of the rule '(x) (x =  $\iota$  x)' which he entertained and rejected (except for the truth-values, the True and the False) earlier in the Grundgesetze (v. I, p. 18, n. 1). He had rejected this rule also in 'A Critical Elucidation of Some Points in E. Schroeder's Vorlesungen ueber die Algebra der Logik,'<sup>1</sup> apparently regarding it as based on a confusion among class-membership, class-inclusion, and the part-whole

<sup>1</sup>Geach, P. and Black, M., editors, Translations from the Philosophical Writings of Gottlob Frege, Second edition (Oxford, 1960), pp.86-106.

relation. In addition, ' $\wedge = \{ \wedge \}$ ' is unwelcome because, as Leonard Linsky and George F. Schumm have shown (Analysis, 32 (1971-72), 507), by applying it to the definitions of zero and one we are able to derive ' $0 = 1$ .' Of course, as Michael Dummett points out (Analysis, 33 (1972-73), 139-40), Frege could escape this conclusion by revising his definitions of zero and one; but a better means of escape--inasmuch as it would not force him to revise his definitions--would be to reject (2) in favor of (1). Similarly, Furth argues (The Basic Laws, [1967 printing], p. 140n.) that Frege has to revise his remarks about the truth-values in the wake of his alteration of Basic Law (V). His conclusion is correct, but the argument is convincing only if (2) rather than (1) is taken to be the new rule. (The best way to prove the conclusion is directly from (3), below.)

Finally, the kind of reasoning which leads from (2) to ' $\wedge = \{ \wedge \}$ ' leads in general to the rule ' $P = P \cup \{ P \}$ ' for all sets P. This is unintuitive, to say the least. On the face of it, if we take an ordinary, well-behaved set, P, and add to it a new number, P itself (for Frege assures us that P does not belong to itself--see (3) below), then we obtain a new set different from P. Frege would surely have avoided the contrary implication if he had seen how to do so. But because he failed to distinguish between (1) and (2) he thought that his consideration of Russell's Paradox provided a good rationale for (2); and he was so anxious to salvage as much of his system as possible that he was prepared to live with whatever unintuitive consequences (2) brought with it, short of outright contradiction. If he had seen that all his reasons in support of (2) really support only (1), and that (2) has nothing in its favor except that it implies (1), he would surely have chosen the latter over the former.

Incidentally, "Frege's way out" is sometimes thought of as the revised criterion of membership:

$$(3) \quad (w \in \hat{A}Fx \cdot \equiv \cdot w \neq \hat{A}Fx \cdot Fw)$$

(see Gg., v. II, p. 264), rather than as either (1) or (2), the latter being criteria of identity for extensions. And the principles needed to get us from (3) to (1) in a simple and direct fashion are absolutely fundamental to set theory: that sets with just the same members are identical, and that concepts under which just the same things fall have identical extensions. On the other hand, no equally well-founded principle will lead us simply and directly on to (2). So on the basis of (3), also, (1) is preferable to (2).

It must be granted that Frege's presentation of his way out is somewhat confused; but I hope to have shown that (1) has a better claim than (2) to the title of Basic Law (V'), and that in any case (1) is a more plausible revision of his system than is (2). Thus criticisms of "Frege's way out" ought to direct themselves to (1) rather than to (2). The arguments of Lesniewski, Geach, and Quine satisfy this requirement; they are based on assumptions (such as (3)) that Frege

certainly did make, and which he would have regarded as following from either version of Basic Law (V'). But the argument of Linsky and Schumm does not satisfy the requirement, and for this reason, if for no other, their claim that 'the falsity of the revised system under its intended interpretation can also be brought out without appeal to the long and rather tedious derivations of Lesniewski et al' (op. cit., 7) remains to be established.

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