WITTGENSTEIN'S EARLY PHILOSOPHY OF MATHEMATICS

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Wittgenstein's remarks in his *Tractatus* on mathematics are quite obscure. Benacerraf and Putnam wrote, "In his *Tractatus Logico-Philosophicus*, Wittgenstein maintained, following Russell and Frege, that mathematics was reducible to logic." On the other hand, Max Black claims, "Wittgenstein does not regard mathematics as reducible to logic, in the manner of Whitehead and Russell." I offer a detailed commentary upon Wittgenstein's remarks, concluding that his views most likely do not follow those of Frege and Russell. I reject a criticism of Wittgenstein presented by Black but find severe shortcomings in the view I take Wittgenstein to be presenting.
Wittgenstein's Early Philosophy of Mathematics

1. Introductory

Wittgenstein's remarks in the *Tractatus* on mathematics are very obscure both in detail and in general import. Paul Benacerraf and Hilary Putnam wrote, "In his *Tractatus Logico-Philosophicus*, Wittgenstein maintained, following Russell and Frege, that mathematics was reducible to logic." On the other hand, Max Black claims, "Wittgenstein does not regard mathematics as reducible to logic, in the manner of Whitehead and Russell." Since such authorities are at loggerheads, I shall in this paper have to spend much time simply indicating how I read the text. This discussion will fall into two parts, corresponding to the traditional way of splitting up the thesis of logicism: (1) The concepts of mathematics can be explicitly defined in terms of logical concepts. (2) The theorems of mathematics can be derived from logical axioms through purely logical deduction.

My own general view, which I shall defend below, is that Wittgenstein was not a logicist in the Frege-Russell sense but that he held the propositions of mathematics to be very like those of logic. To present my view in detail, however, I must start with a discussion of operations.

2. Operations

The main discussion of operations in the *Tractatus* is in


4From Rudolf Carnap's "The Logicist Foundations of Mathematics" in *The Philosophy of Mathematics*, p. 31. These two parts of the logicist thesis are not independent; (2) could not be true if (1) were false.
the 5.2's. Starting from the assertion that the "structures of propositions stand in internal relations to one another" (5.2), which is the linguistic reflection of the fact that possible situations stand in internal relations to one another (4.125), Wittgenstein says, "In order to give prominence to these internal relations we can adopt the following mode of expression: we can represent a proposition as the result of an operation that produces it out of other propositions (which are the bases of the operation)." (5.21) Then, in what seems to be the closest to a definition of 'operation' in the Tractatus, Wittgenstein adds, "The operation is what has to be done to the one proposition in order to make the other out of it." (5.23)

Two comments are necessary. First, although Wittgenstein seems to be at pains in some of his remarks to emphasize that operations connect propositions (in addition to those quoted above, see 5.233), he also writes, "The internal relation by which a series is ordered is equivalent to the operation that produces one term from another." (5.232) An example of such a series (a series of forms, as he also calls it) is the number series (4.1252), and signs for numbers are not propositions. Secondly, since N, the operation of joint negation, is the paradigm instance of an operation (5.5, 5.502) and since N operates on an arbitrary number of propositions (5.503), one must overlook the implication in 5.23 (and 6.01) that an operation has only a single proposition as its basis.

Since the number series is a formal series, it is important to introduce Wittgenstein's notation for an arbitrary term in such a series; he calls it a notation for the general term of a series of forms (5.2522). If O is an operation then the general term of the series of forms, a O'a, O'0'a, . . . is indicated by '[a, x, O'x]', in which: (i) 'a' represents the initial term of the series, (ii) 'x' indicates any term selected arbitrarily from the series, and (iii) '0'x' is the term that immediately follows 'x' in the series. Wittgenstein called the bracketed expression a variable.

3. The Concepts of Mathematics

The main discussion of the concepts of mathematics in the Tractatus is in the 6.0's. Wittgenstein has already claimed that all propositions can be generated by applying the operation N to the elementary propositions. This is what all propositions have in common; hence the general form of a proposition can be represented in the notation introduced in the
last section as

\[ \overline{\bar{P}}, \overline{\bar{\xi}}, N(\bar{\xi}) \]

where \( P \) must be the set of all elementary propositions (6; 3.341. 5.471-5. 4.72).\(^5\)

Once we are given the general form of a proposition, then also we are given "the general form according to which one proposition can be generated out of another by means of an operation" (6.002), and so Wittgenstein introduced "the general form of an operation \( 0'( ) \)" as follows:\(^6\)

\[ \overline{\bar{\xi}}, N(\bar{\xi}) \] \( (\overline{\bar{\eta}}) \) \( (=\bar{\eta}, \overline{\bar{\xi}}, N(\bar{\xi}) \)).

It is difficult, however, to see just what is being generalized here. \( 0 \) seems to be no more than \( N \), the operation of joint negation. But this, claims Wittgenstein, is how we arrive at numbers.

An operation may be applied to its own results, which Wittgenstein termed "successive applications" of that operation (5.2521). In 6.02, he considered the formal series generated by applying the general form of an operation successively, starting from an arbitrary base. That would be written:

\[(1) x, 0'x, 0'0'x, 0'0'0'x, \ldots . \]

He presented the following definitions:

\[(2)\]

\[x = 0^0_1x \quad \text{Def.},\]

\[0'0^n_1x = 0^{n+1}_1x \quad \text{Def.},\]

and then re-wrote (1) as:

\[(1') 0^0_1x, 0^{0+1}_1x, 0^{0+1+1}_1x, 0^{0+1+1+1}_1x, \ldots . \]

Then he presented a final set of definitions:

\(^5\)As Black points out (Companion, p. 312), Wittgenstein has introduced the bar notation only in connection with variables (5.501).

\(^6\)I use '0' where Wittgenstein uses '\( \Omega \)'.

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(3) \[ 0 + 1 = 1 \] \hspace{1cm} \text{Def.},

\[ 0 + 1 + 1 = 2 \] \hspace{1cm} \text{Def.},

\[ 0 + 1 + 1 + 1 = 3 \] \hspace{1cm} \text{Def., (and so on).}

Presumably, then, (1') could be re-written as:

\[ (1'') 0^0x, 0^1x, 0^2x, 0^3x, \ldots , \]

but Wittgenstein did not explicitly draw this conclusion.
From these definitions, Wittgenstein did conclude, "A number is the exponent of an operation." (6.021)

One preliminary comment on the above is in order before proceeding to the rest of the 6.0's. From the realistic standpoint of Frege, all Wittgenstein has done so far is to introduce numerals. Did Wittgenstein identify numbers with numerals? If he did, why did he not mention and try to dispose of the formidable barrage of criticism that Frege aimed at that position? If he did not, what then are the numbers? Since there are no remarks in the Tractatus addressed directly to these questions, I shall have to try to answer them, later in this paper, on the basis of whatever indirect evidence I can muster.

The remaining remarks of the 6.0's are very obscure -- obscure in intent if not in meaning. Wittgenstein wrote, "The concept of number is the variable number." (6.022) This is so because the concept of number is identified with "what is common to all numbers" (6.022), which is identified with "the general form of a number" (6.022), which in turn is given as

\[ [0, \xi, \xi + 1] \] \hspace{1cm} (6.03);\footnote{See Translations from the Philosophical Writings of Gottlob Frege, eds. Peter Geach and Max Black (Oxford, 1960), pp. 182-223.}

and he has already called this notation a variable (see section 2.).

The first difficulty with these remarks is how to reconcile them with 6.021. If a number is really the exponent of an integer, but he had no systematic distinction. \footnote{Wittgenstein actually wrote in 6.03 that \([0, \xi, \xi + 1]\) is the form of a \textit{whole number} (\textit{gansen Zahl}) or integer, but he had no systematic distinction.}
operation, then surely the general form of a number is that which is given in 6.02:

\[ 0^0 \cdot x, 0^n \cdot x, 0^{n+1} \cdot x \].

In this light, 6.03 seems just wrong. The second difficulty is simply that, since Wittgenstein made minimal use of the concept of number and the general form of a number, it is unclear why he introduced them at all.

The final remark of the 6.0's is 6.031: "The theory of classes is completely superfluous in mathematics. This is connected with the fact that the generality required in mathematics is not accidental generality." The first assertion presumably reflects Wittgenstein's belief that he had just exhibited a way to introduce numbers without relying on set theory, as Whitehead-Russell and Zermelo had done. The second remark must be connected with Wittgenstein's criticisms of Whitehead and Russell's Axiom of Infinity (5.535) and Axiom of Reducibility (6.1232-6.1233), especially the latter. He claimed that these axioms, essential to the derivation of the propositions of mathematics in *Principia Mathematica*, could be more than accidentally true. I am not able even to hazard a guess, however, as to the connection Wittgenstein saw between the first and second of these remarks.

4. A Criticism Considered

"Russell," writes Black, "... bore the applications of arithmetic to counting in mind. Wittgenstein seems to have made no provision for such application." Although there is much yet to be said about the Tractarian view of mathematics, I believe that Black's criticism, which pertains solely to Wittgenstein's numbers, can profitably be discussed now.

What conditions must be met in order that one's numbers, in some reconstruction of number theory, be applicable to counting? Quine, fortunately, has given a clear account of the matter:

\[ \text{Companion, p. 314.} \]
The condition upon all acceptable explications of number (that is, of the natural numbers 0, 1, 2, . . .) can be put succinctly: any progression -- i.e., any infinite series each of whose members has only finitely many precursors -- will do nicely. Russell once held that a further condition had to be met, to the effect that there be a way of applying one's would-be numbers to the measurement of multiplicity: a way of saying that (1) There are \( n \) objects \( x \) such that \( Fx \). This, however, was a mistake; any progression can be fitted to that further condition. For, (1) can be paraphrased as saying that the numbers less than \( n \) admit of correlation with the objects \( x \) such that \( Fx \). This requires that our apparatus include enough of the elementary theory of relations for talk of correlation, or one-one relation; but it requires nothing special about numbers, except that they form a progression.\[10\]

The series of numbers (or numerals) introduced above by Wittgenstein is a progression in Quine's sense and so would be an "acceptable explication of number" if a way can be found to express Quine's (1). Wittgenstein, however, has dismissed set theory from his "apparatus", so he can have no theory of extensional relations; and while it might well be argued that a theory of intensional relations would be compatible with all that Wittgenstein has written in the Tractatus, it is clear that no such theory is presented there. How, then, can Wittgenstein express (1)?

The answer, I believe, is provided by 5.5321. Back at 5.53 Wittgenstein introduced the following convention: "Identity of object I express by identity of sign, and not by using a sign for identity. Difference of objects I express by difference of signs." Using this convention, he then remarks at 5.5321, "And the proposition 'Only one \( x \) satisfies \( f( ) \)', will read '(Ex).fx: ~(Ex,y).fx.fy'." Following this lead (and modernizing the notation slightly), Quine's (1) can be expressed in Wittgenstein's correct conceptual notation as

\[
(1^*) (Ex_1, x_2, \ldots, x_n) (Fx_1 \& Fx_2 \& \ldots \& Fx_n) \&
\]

\[10\] Introduction to Mathematical Philosophy, p. 10

\[11\] Word and Object (Cambridge, Massachusetts; 1964), pp.262-263.
where the subscripts on the variables may be in the expanded form \(0+1\), \(0+1+1\), etc. Thus I believe that Wittgenstein can meet all Quine's requirements and hence escape Black's criticism.

5. The Propositions of Mathematics

The main discussion of the propositions of mathematics in the Tractatus is in the 6.2's. The parallels between the points made there about the propositions of mathematics and those made in the 6.1's about the propositions of logic are quite striking. Here is a summary of the 6.1's: (1) The propositions of logic are tautologies; they say nothing. (2) The propositions of logic show the formal (or logical) properties of language and of the world. (3) A logical truth can be recognized by calculation from the symbol alone. Proof in logic is superfluous. (4) The tautologies are, in fact, not needed, since what they show could (in a proper notation) be seen from the constituent propositions themselves. Now the 6.2's: (1') The propositions of mathematics are equations; they do not express thoughts. (2') The propositions of mathematics, like the tautologies of logic, show the logic of the world. (3') the "correctness" of a mathematical proposition can be perceived without comparing it to the facts. (4') Mathematical equations are not needed, since what they show must be evident from the expressions themselves. These similarities are what I had in mind when I wrote above that in Wittgenstein's view mathematics was quite like logic. Let us examine them now in detail.

As for (1) and (1'), a tautology says nothing, even though it is genuinely part of the logically correct language, because it agrees with all the truth-possibilities of its elementary propositions; it excludes nothing. An equation says nothing because it is not part of the logically correct language at all (see 5.533). Black's remarks that equations are "part of the symbolism" are wrong.12

Wittgenstein throws some light on (2) in 6.124, in which he claims that the propositions of logic presuppose that there are objects and atomic facts. No such illumination is

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12 Companion, pp. 341, 380. I have argued this at length in Chapter III of my unpubl. diss. (Brandeis, 1972) "Frege and Wittgenstein on Identity, Logic and Number".
provided for (2'). He does say that equations show that "the two expressions connected by the sign of equality have the same meaning" (6.232), but surely this is a poor candidate for the logic of the world. Moreover, if Wittgenstein did provide some convincing illustration of (2'), he would then have to reconcile it with 6.232. I find this remark of Wittgenstein's (6.22) very mysterious.

Concerning (3) it is important to emphasize that it was supposed by Wittgenstein to state "the peculiar mark of logical propositions" (6.113). "The whole philosophy of logic" is contained in the fact that a logical truth can be recognized as such without having to compare it to reality (6.113). That he supposed equations to have this property too seems sufficient grounds for asserting that on Wittgenstein's views the similarities between the propositions of logic and mathematics are far more significant than their differences.

That equations do have this property is the point of 6.232: "And the possibility of proving the propositions of mathematics means simply that their correctness can be perceived without its being necessary that what they express should itself be compared with the facts in order to determine its correctness." So this is what proof of an equation comes to -- its correctness can be seen by an examination of the signs which comprise it.13 But how is this correctness perceived? On Wittgenstein's general view of identity-statements, the truth of 'a=b' is found by examining the elementary propositions of the correct conceptual notation or perspicuous language to see whether for each such proposition containing 'a' there is another proposition containing 'b' in the corresponding position (and vice versa).14 This observation, I would conjecture, explains at least part of what Wittgenstein meant in 6.233: "The question whether intuition is needed for the solution of mathematical problems must be given the answer that in this case language itself provides the necessary intuition."

13 I discuss another interpretation of 6.232 in the next section.

14 This point I have discussed more fully in my dissertation (in Chapter III, Section 5), which is cited in footnote 12.
I must comment here on some remarks in the Tractatus -- those remarks on proof in mathematics in 6.24 and 6.241 -- which might seem to contradict what I have just said. Wittgenstein writes in 6.2341: "It is the essential characteristic of mathematical method that it employs equations. Indeed it is a consequence of this method that every proposition of mathematics must be obviously true." 'Obviously true' cannot, as Black correctly points out, mean self-evident, a notion which Wittgenstein scorns (6.1271). It must mean something like: not in need of any appeal to facts in order to determine their truth or falsity. Thus while all the propositions of mathematics must be "obvious" in this sense, they need not be obvious in the ordinary sense of easily recognizable; and I take 6.24 and 6.241 to be saying that by "the method of substitution" one can start with obvious (in the ordinary sense) equations and proceed to generate unobvious ones, thus facilitating recognition of these less evident ones, but all these equations are nonetheless obvious in Wittgenstein's special sense. This interpretation provides yet another parallel between the 6.2's and the 6.1's.16

As for (4), Wittgenstein writes that "in a suitable notation" tautologies would be superfluous, for we could "recognize the formal properties of propositions by mere inspection of the propositions themselves". (6.122) For example, that

\[(p \supset q) \land p \supset q\]

is a tautology shows that 'q' follows from '(p \supset q) \land p', but we could see this from the propositions themselves (6.1221). To parallel this, Wittgenstein wrote: "But the essential point about an equation is that it is not necessary in order to show that the two expressions connected by the sign of equality have the same meaning, since this can be seen from the two expressions themselves." (6.232)

There is one important observation to be made about (4) and (4'). What tautologies show (at least, part of what they show; see 6.1201 for another example) and what equations show

15Companion, p.343.
16One would expect the "method of substitution" (6.24) to consist of terms in equations being replaced by other terms with which they had previously been equated, but the sample 'proof' of 6.241 depends on various manipulations of the operation-sign, for which no provision has been made.
is that a certain sort of inference is permissible. The former shows (sometimes) that a certain proposition follows from certain others, and the latter shows that a certain substitution can be made. This similarity seems to be what Wittgenstein had in mind in 6.2: "Mathematics is a logical method." (See also 6.234.) But this similarity does not provide sufficient grounds for rendering 6.2 and 6.234 as "Mathematics is a part of logic" as B.V. McGuinness did.17

6. A Criticism

The great logicists, Frege and the authors of Principia Mathematica, did not merely provide explications of the natural numbers, an accomplishment which the quote from Quine in Section 3 should convince us is relatively slight; they also showed what assumptions or axioms were necessary in order to derive the propositions of mathematics by purely logical inference rules and how these derivations would be constructed. It is well-known that they failed to establish the second half of the thesis of logicism -- in Frege's case because his system was inconsistent, in Whitehead and Russell's because the Axiom of Reducibility was not a purely logical axiom. It is time now to see how solid a foundation Wittgenstein gives to the propositions of mathematics.

The crucial doctrine of Wittgenstein's is the one I have labelled (3'), the view that the "correctness" of a mathematical proposition can be perceived without comparing it to the facts. In place of the formally precise notion of proof of Frege and Russell, Wittgenstein substituted his nebulous doctrine of showing by a correct notation. Of course, there are deep obscurities about the idea of a correct notation, especially since Wittgenstein gives no clear description of the process of analysis which leads one to it; but let us put this problem aside and return to one I introduced in Section 3. In the 6.0's is Wittgenstein introducing numbers or numerals?

Although Wittgenstein writes that "A number is the exponent of an operation" (6.021), I claim that his exponents of operations (i.e., his numbers) are symbols; hence, from a Fregean point of view, they are merely numerals or names of numbers -- not the numbers themselves. An example of a mathematical proposition is '2 x 2 = 4' (6.241). This implies that the '1', '2', '3', etc. introduced in 6.02 are among the symbols that flank '=' in the equations -- a claim which is hardly surprising. But these expressions are spoken of as having meaning (Bedeutung) in 6.232 and 4.241; and this is absolutely necessary for maintaining (3'), since it is the identity of meaning of two expressions that is supposed to show itself in the symbols themselves and that obviates any investigation of reality.

But what are the meanings of these numerals? Are they objects, which are spoken of as the meanings of names (3.203)? If they are, then it would be clear -- clear for the Tractatus, at least -- how language could show the truth of the propositions of mathematics. There is much reason, however, to hold that they are not. First of all in the 4.12's, Wittgenstein introduced the notion of a formal concept. He wrote that 'object', 'complex', 'fact', 'function', and 'number' signify formal concepts (4.1272) and that it can not be said but only shown that something falls under a formal concept. Since Wittgenstein spoke of 'object' and 'number' as signs for different formal concepts and since the implication of that whole section is that not merely different entities, but entities of different types or categories, fall under the various formal concepts mentioned there, one must conclude that numbers are not objects.

Secondly, if Wittgenstein believed that the meanings of numerals were objects, then he would either have to admit the possibility that his arithmetic might have only a finite number of numerals, which is absurd, or he would have to hold it to be evident that there were an infinite number of objects. Wittgenstein wrote, however: "Elementary propositions consist of names. Since, however, we are unable to give the numbers of names with different meanings, we are also unable to give the composition of elementary propositions." (5.55; Emphasis added). 'Give' in this context clearly means give a priori; and since the meaning of a name is an object (3.203), Wittgenstein implies that it is not possible to determine a priori the number of objects. (See also 5.557.)
One must conclude from this that the meaning of the numerals introduced in the 6.0's are not objects but (simply) numbers, but there seems to be no discussion of these numbers in the Tractatus at all! And even more important, there is barely a hint in the Tractatus as to how two expressions could be seen (as in 6.232) to mean the same number, and so (3') would be left largely unsupported. Perhaps it is plausible that one just could see that 1 + 2 = 3 (i.e. that (0+1) + (0+1+1) = 0+1+1+1, when the numerals are written out fully), but to handle equations involving large numbers some procedure must be given for comparing one side of the equation with the other. Transformation of one side into the other through substitutions licensed by simpler equations possibly -- but this begins to look like proof.

Perhaps 6.232 should be construed in this broad fashion. One could then read Wittgenstein as introducing a set of axioms in 6.02 and describing a method of proof in 6.241; and then 6.2321 may be read as saying that since equations can be proved, they need not be compared with facts. This interpretation of Wittgenstein is closer to Frege and to Whitehead and Russell than any of the others discussed above, giving it at least a familiar feel; but it does not merely modify, it flatly contradicts, 6.232 because it is now the proof of an equation rather than "the two expressions themselves" that shows that the two expressions connected by the sign of equality have the same meaning. To interpret the text in such a way that the primary commentary to a proposition (6.2321) contradicts the proposition on which it comments (6.232) seems to me indefensible. Moreover, since the propositions of mathematics play an important role (6.211) and since on this interpretation they must be proved rather than seen, Wittgenstein can no longer say that the identity sign is "not an essential constituent of conceptual notation". (5.533) Thus Wittgenstein's account of the equations of mathematics, on this interpretation, would be incompatible with his general account of identities.

7. Conclusion

None of these alternatives are very satisfying. From the standpoint of philosophy of mathematics, ignoring the rest of Wittgenstein's system, the last is the most plausible, but perhaps that plausibility ultimately rests on its similarity to more familiar views. If this is Wittgenstein's view, then the method of proof is left so vague that it is scarcely possible to evaluate it; and even if all the details could be filled in, one should be under no illusion about
what is accomplished. Hilbert wrote that "the science of mathematics is by no means exhausted by numerical equations and it cannot be reduced to these alone."18 Ramsey said that Wittgenstein's view of the propositions of mathematics as equations "is obviously a ridiculously narrow view of mathematics, and confines it to simple arithmetic..."19 Wittgenstein's view, could some way be found to make it work, could not even account for the principle of mathematical induction, much less the real numbers. At best, it is scarcely the beginning of a philosophy of mathematics.

It should not be surprising, then, that so sketchy a position should evoke the contradictory comments quoted at the beginning of this chapter. Since I basically side with Black on this matter, though I disagree with him about important details, I shall give him the last word: "So, Wittgenstein does not regard mathematics reducible to logic, in the manner of Whitehead and Russell. On the other hand, what Wittgenstein has already called 'the peculiar mark of logical propositions' (6.113a), the possibility of perceiving their validity by inspection of the signs that express them, without reference to facts or any other external source of evidence, is equally characteristic of mathematical equations (6.2321). In view of this, the distinction between mathematics and logic must seem somewhat arbitrary."20

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