

Kant's Theory of Mathematics Revisited

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In a number of earlier publications I have outlined an interpretation of Kant's theories of the mathematical method, space and time, and the analytic-synthetic distinction (at least insofar as this distinction applies within mathematics).¹ I have also tried to show how these views of Kant's entered into the structure of his philosophical thought. Since Kant held that the gist of the mathematical method lies in the use of constructions, i.e., in the use of intuitions to represent general concepts, I have also presented an analysis of the meaning of the term "intuition" (or, strictly speaking, of its counterparts *intuitus* and *Anschauung*) in Kant. This interpretation has prompted a few relatively detailed criticisms.² One of the purposes of this article is to return to my interpretation in the light of the criticisms. Naturally, one focal point of my paper will be a discussion of the main texts on which my critics base their allegations. Even more important is the view we have to take of the historical background of Kant's theory.

I cannot here recount adequately my interpretation but must be content with a summary.³ The interpretational basis of my theory is, in a nutshell, as follows: By intuition (*Anschauung*), Kant meant a representative ("*Vorstellung*") of a particular entity in the human mind. By *construction*, Kant meant the introduction of such a particular to instantiate a general concept. The gist of the mathematical method *apud* Kant was the use of such constructions (a modern logician would say "the use of rules of instantiation"). A mathematical argument is synthetic if it involves the use of "auxiliary constructions," i.e., the introduction of new particulars over and above those given in the conditions of the argument (sometimes given in the premises and sometimes given in the premises *or* in the purported conclusion). A mathematical truth is synthetic if it can be established only by such synthetic arguments.

Among the consequences of the interpretation are the following:

1. There is nothing "intuitive" in the basic force of the concept of intuition in Kant. Insofar as there is a reference to imagination or sensation contained in Kant's concept, it is supposed to be an

outcome of his arguments, not a *presupposition* of those arguments.

2. The immediacy of intuitions in relation to their objects is merely a corollary to their particularity.

3. This immediacy of intuitions has no bearing on their value as helping to establish any a priori truths, let alone on their alleged status as a source of such truths.

4. The use of constructions and intuitions in a mathematical, e.g., geometrical, argument does not mean appeal to what in our contemporary usage would be called mathematical (geometrical) "intuition." It merely means using particular representatives of general concepts, i.e., instantiation rules, in a mathematical argument.

5. What Kant says of mathematics pertains more to contemporary first-order logic, whose mainstays are precisely instantiation rules, than to what we twentieth-century philosophers would classify as specifically mathematical modes of reasoning.

6. What makes mathematical truths synthetic is brought out by the mode of argument used in establishing them.

7. More specifically, the intuitive (constructive, synthetical) steps of a mathematical argument are, according to Kant, firmly within the axiomatic and deductive proofs, not in collateral appeals to intuition (in our present-day sense) or in the nature of the axioms of a mathematical theory.

In spite of strong textual support for my interpretation, mistakes abound in the literature. For instance, the two latest major German encyclopedic dictionaries of philosophy botch up their respective articles on *Anschauung* as far as Kant is concerned.⁴ Perhaps more important, several scholars are still reluctant to adopt my interpretation, as illustrated by the specific criticisms they have prompted. I will not try to present a detailed answer to all the criticisms. It seems to me more constructive to discuss further the direct problems of Kantian interpretation. In this respect I have found extant discussions of my views keenly disappointing, not to say frustrating. The discussions have centered on the interpretation of particular passages. Next to no attention has been paid to the overall picture of Kant's thinking about mathematics in its historical setting or to my view of the role of Kant's theory of space, time, and mathematics (including the whole of his transcendental aesthetics) within the structure of his philosophical system. It seems to me that this exemplifies a much more widespread fault with philosophers' discussions of historical matters. Whether a philosopher likes it or

not, the way he interprets a major character in the history of philosophy depends crucially on the view he takes of that philosopher's historical position and problem situation. If these questions are not raised explicitly, the philosopher in question is in effect relying on unexamined and usually uncritically adopted received views, not to say prejudices. Yet all too frequently these questions are not even raised, and all of the discussion centers on the interpretation of particular passages taken out of context. If specific examples are needed, it seems to me that there is no hope whatever of understanding Aristotle's discussion of future contingents without first understanding his general assumptions concerning time, necessity, truth, and chance.⁵

Likewise, it may be interesting and important to compare Kant's philosophy of mathematics with such contemporary problems as intuitionism or formalism, but only if one realizes at the same time that such problems simply were not within his intellectual horizon. What paradigms of mathematical practice did Kant know? There is one overwhelming answer. If we look at what Kant read as a student and as a mature philosopher, what he thought, and what he dabbled in *qua* mathematical amateur, the answer is crystal clear. Kant's view on what mathematicians "really" do is modeled on Euclid's *Elements* and its eighteenth-century variants (one probable reason for the neglect of the role of Euclid in providing the paradigm of mathematics for Kant is philosophers' ignorance of the extent to which Euclid's *Elements* dominated elementary and secondary mathematics education in Kant's time).⁶ Kant's only extensive attempts to do something on his own in mathematics were vain efforts to prove Euclid's fifth postulate.

Now Euclid's procedure offers an obvious and clear model of Kant's views on the mathematical method and on the concepts of intuition and construction. All the different features of Kant's philosophy of mathematics mentioned above as a part of my interpretation are strongly and immediately suggested by Euclid's procedure.⁷ Hence my interpretation, unlike its rivals, is defensible by reference to the letter of Kantian texts. Moreover, when viewed in the right perspective, it turns out that my interpretation ascribes to Kant the most natural view of mathematics that anyone in Kant's historical situation (and with Kant's limitations) could possibly have adopted. However one is inclined to read individual passages in Kant's texts, in the last analysis they have to be judged against the whole of his historical situation. This is what my interpretation is

calculated to do. I have not seen any similar attempt made on behalf of its competitors.

How is it that Euclid actually proceeds in presenting and establishing a proposition? He first presents a general proposition. To use Kant's favorite example as our illustration, he formulates proposition 32 of book 1 by saying that "in any triangle . . . the three interior angles of the triangle are equal to two right angles."

Euclid, however, never does anything directly on the basis of this general enunciation or *protasis*. He always goes on to apply (as we are tempted to say) the enunciation to a special case (some particular figure). In our example, he says: "Let ABC be a triangle. . . . I say that the three interior angles ABC , BCA , and CAB are equal to two right angles." This part of a Euclidean proposition was called the *ekthesis*. The same term was used by Aristotle in his logical work for what in modern terms is to all practical purposes instantiation. The received English terms for this part of an Euclidean proposition are *exposition* and *setting-out*.

If we look away from the sometime part of an Euclidean proposition called *diorismos*, we can say that *ekthesis* is followed by the *auxiliary construction* or *preparation*. In it some new geometrical objects are introduced into the argument. In proposition 32 this part reads, "For let CE be drawn through the point C parallel to the straight line AB ." Such introductions of new geometrical objects are justified by the postulates of Euclid's system or by solutions to earlier problems, in our sample case by proposition 31 of book 1.

The auxiliary construction is followed by the proof (proof proper), the *apodeixis*. This is the only part in which an actual demonstrative argument is carried out. Its several steps are based on the assumptions Euclid calls *Common Notions*, theorems proved earlier, plus the properties ascribed to the geometrical objects in question on the basis of their "construction," i.e., on the basis of what general properties they were introduced to instantiate in the *ekthesis* and in the auxiliary construction. This is illustrated by proposition 32. In its *apodeixis* part, Euclid appeals to earlier theorems, to common notions, and to the general characteristics the different parts of the relevant diagram were introduced to instantiate ("Again, since AB is parallel to CE . . .").

Here we can see what Kant's ideas of the mathematical method are modeled on and what they amount to. When he says that mathematics is based on the use of intuitions, after having defined intuitions as particular *Vorstellungen*, what he has in mind is Euclid's use of what looks like particular cases. Another variant of this

Kantian jargon is to say that the mathematical method is based on the use of *constructions*, which are defined by him as introductions of particular entities to instantiate general concepts. The role they play in Euclid's system of geometry is essentially that of the instantiating free symbols used in modern logic, especially in the so-called natural-deduction methods, notwithstanding the frequently repeated myth that Euclid needed them in order to appeal tacitly to geometrical "intuition" (in the twentieth-century sense of the term). Admittedly, Euclid occasionally takes steps in his arguments which can only be explained as being based on geometrical "intuition." In so doing, however, he was probably violating his own principles, and he was beyond any doubt violating Kant's principles. As he formulates them, if a geometer "is to know anything with a priori certainty he must not ascribe to the figure anything save what necessarily follows from what he has himself set into it in accordance with his concept" (Bxiii). Can you rule out appeals to geometrical "intuition" more explicitly?

Why, then, does Kant think that the use of constructions (instantiations) is useful (indeed, indispensable) in mathematics? The reason is that he realized, without being at all clear about the logical basis of the phenomenon, that *certain arguments simply cannot be carried out without the use of auxiliary constructions*. This is the datum on which Kant bases his theories of mathematics, space, and time. In spite of the fact that Kant makes the point forcefully and unmistakably, especially in A712–27/B741–55, this fundamental feature of his overall argument has been completely missed by the philosophers who have recently discussed Kant's philosophy of mathematics. It was noted by C. S. Peirce and made the basis of one of the central tenets in the philosophy of logic, but Peirce's insight too was completely overlooked until I called attention to it.⁸

This observation has several important consequences. For one thing it means that the nontrivial (synthetic) element in geometrical reasoning lies in the auxiliary constructions. Accordingly, it is in those constructions that Kant locates the synthetic element in mathematical (for us, logical) reasoning. In contrast, the "common notions" on which *apodeixis* ultimately rests were taken by Kant to be analytic. As he puts it, "Some few fundamental propositions presupposed by the geometrician are, indeed, analytic, and rest on the principle of contradiction. But, as identical propositions, they serve only as links in the chain of method, and not as principles, for instance, $a = a$; the whole is equal to itself; or $(a + b) > a$, that is, the whole is greater than its part" (B16–17). The last of these examples is

precisely Euclid's fifth Common Notion. Further evidence is found in A164/B204.

For another thing, we can now see that the *explanandum* of Kant's transcendental aesthetics contains no reference to any specially intimate relation between an intuition and its object. Nor is there any trace here of the idea that the reason an intuition in the Kantian sense can give us new information is its especially close relation to what it stands for (*vorstellt*). There was in Kant's background a very strong assumption that the intuitivity of a *Vorstellung* meant an immediacy in its representational role. Kant, we can now see, reinterpreted this immediacy so that it amounted completely to individuality (particularity). It is simply and solely the role of Kant's so-called intuitions *as instantiating terms* (and hence as particulars corresponding to a concept) that helps us gain information by their means in mathematical proofs.

But, it may be objected, surely the Euclidean paradigm covers only geometry, not arithmetic or algebra. Furthermore, limited though Kant's mathematical knowledge may have been, he could not have been oblivious of such developments as Descartes's "analytical" geometry.

In reality, Descartes's geometry offers crucial further evidence for my interpretation. If we note what Descartes emphasizes in his own new geometry, we see that it supplies to Kant the missing ingredient which enabled him to extend concepts based on the Euclidean model to other branches of mathematics. In Descartes's mind and in his readers' first impressions, the great novelty of his treatment of geometry was not the use of coordinates. They slip into his book only as helpmeets in dealing with specific problems. As is shown by the very first sentences of *La Geometrie*, Descartes sees the essence of his mathematical method in a systematic and comprehensive *analogy between geometrical constructions and algebraic operations*.⁹ It was precisely by means of this analogy that Kant was able to think of the concepts he had first formulated by reference to geometrical constructions as being applicable to all mathematics. Indeed, it is in Kant's references to simple arithmetical equations like $7 + 5 = 12$, that his adherence to the Euclidean model becomes most conspicuously clear. He calls them "indemonstrable" and "immediately certain" (A163–64/B204–205). Yet he describes a process by means of which these equations are ascertained (B15–16) and even says that their syntheticity is seen more clearly if we take greater numbers.

It is nearly incredible that most interpreters of Kant still

apparently maintain in the teeth of such passages that according to Kant the synthetic nature of mathematical reasoning is due to appeals to "intuition." Clearly it is not. But to what, then, is it due? What does Kant mean by calling $7 + 5 = 12$ "indemonstrable"? The Cartesian analogy supplies a ready answer. Such equations are "indemonstrable" in the literal sense that no counterpart to the Euclidean demonstration or *apodeixis* is needed, Kant thinks, in establishing them. It suffices to carry out *ekthesis* and the auxiliary construction, which for Kant means the addition of one number to the other unit by unit (cf. B15–16). And, of course, it is the use of auxiliary construction that makes an argument synthetic according to Kant, and, of course, the need of actually carrying out the addition is seen more keenly in larger numbers. As I have pointed out earlier, Kant to all practical purposes affirms this way of looking at what he is saying in his letter of November 25, 1788, to Johann Schultz. The reason he calls such equations as $7 + 5 = 12$ "immediately certain practical judgments" is that they need no "resolution" and no "proof," these being the current labels for standard parts of geometrical arguments. "Resolution" was in fact another name of the *diorismos* part (of problems rather than theorems) which was mentioned but not described above.

The evidence which I have briefly (and partly) surveyed shows convincingly that it was the use of auxiliary constructions (plus, possibly, of *ekthesis*—Kant is not clear on this subsidiary point) that makes many mathematical (for us, logical) arguments synthetic and that contains the gist of the mathematical method. It would be extremely interesting to see how Kant uses this idea in the rest of philosophy. This task is too large to be undertaken here, however.

Instead, let me simply sharpen the problem an interpreter is facing here. What we have found is (as was mentioned above) that the synthetic element in mathematical reasoning is squarely within the framework of the Euclidean axiomatic and deductive treatment of geometry (and mathematics more generally). It lies in the use of procedures which are closely related to the instantiation rules of modern logic. This is the *datum* which Kant's Transcendental Aesthetics is calculated to explain, not the alleged use of spatial and temporal imagination in mathematical arguments. Whoever does not appreciate this fact has not understood what Kant's Transcendental Aesthetics is all about.

Here I shall not analyze the ways in which Kant sought to account for this remarkable feature of the mathematical practice of his time and the logical practice of our days. Perhaps it is historically

understandable that his thought took the turn it did even when this purely axiomatic character of Kantian use of intuitions is realized. After all, Alexander the Commentator had explained the use of instantiation or *ekthesis* in Aristotelian logic in a way distinctly reminiscent of the Transcendental Aesthetics. Such an *ekthesis* is according to Alexander comparable to an appeal to sense perception. Be this as it may, however, here I will merely try to clear the underbrush further and eliminate one more objection to locating the synthetic element in mathematical reasoning in *ekthesis* and in auxiliary constructions, i.e., in certain parts of mathematical *arguments*. As has been pointed out repeatedly by Ernst Cassirer, among others, and recently by Gordon Brittan,¹⁰ there is a passage in Kant which *prima facie* suggests a different view, a view which these two scholars have in fact adopted. According to this view, what makes mathematical truths synthetic is not any feature of mathematical *arguments* but the nature of mathematical *axioms*. To defend my own interpretation, I must therefore discuss this passage. It runs as follows:

All mathematical judgments, without exception, are synthetic. This fact . . . has hitherto escaped the notice of those who are engaged in the analysis of human reasons, and is, indeed, directly opposed to their conjectures. For as it was found that all mathematical inferences (Schlüsse) proceed in accordance with the principle of contradiction (which the nature of all apodeictic certainty requires), it was supposed that the fundamental propositions of the science can themselves be known to be true through that principle. This is an erroneous view. For though a synthetic proposition can indeed be discerned in accordance with the principle of contradiction, this can only be if another synthetic proposition is presupposed, and if it can then be apprehended as following from this other proposition; it can never be so discerned in and by itself. [B14]

The first thing to be noted about this passage is that Kant is *not* saying that mathematical truths *always* get their syntheticity from the syntheticity of earlier theorems (and ultimately from axioms or postulates) from which they can *always* be deduced analytically. All he is saying is that this *can* happen and that the only way in which a synthetic proposition can thus be deduced analytically is from another synthetic proposition. This possibility is emphasized by Kant as one of the reasons for his opponents' mistakes rather than as an ingredient of his own constructive theory. Moreover, Kant could

not conceivably be saying in the quoted passage that the syntheticity of synthetic mathematical truths can in all parts of mathematics be traced to the syntheticity of its axioms, for the simple reason, that according to Kant, arithmetic has no axioms (A163–65/B204–205), only “fundamental propositions” which are established one by one without being reducible to any general axioms. From this it follows that not all the “fundamental propositions” (*Grundsätze*) Kant mentions in the quoted passage are axioms. This is confirmed in B16, where Kant says, after having discussed simple arithmetical equations like $7 + 5 = 12$, “just as little is any *fundamental proposition* [my italics] of pure geometry analytic.” And, as we all know, Kant goes through a whole song and dance to spell out how it is that our way of establishing equations like $7 + 5 = 12$ makes these propositions of arithmetic synthetic.

Moreover, Kant’s remarks are addressed not to the status of all and sundry mathematical truths but only to those fundamental propositions which earlier analysts of human reason had mistakenly thought they could prove analytically. All he is saying is that even these particular allegedly analytical *Grundsätze* are based on some more fundamental propositions, which are synthetic. The obvious candidates for this role of earlier analysts of mathematical foundations are Leibniz and Wolff. Indeed, Kant’s comments on $7 + 5 = 12$ on the very following page can very well be thought of as his reply to Leibniz’s claim that such arithmetical propositions can be proved logically. But if so, Kant’s statements have no bearing on mathematical truths in general, more specifically, no bearing on the question whether the syntheticity of most mathematical truths is an inheritance from the syntheticity of mathematical axioms and postulates or whether it is due to the way they are proved. In sum, contrary to popular misconceptions, Kant never says that the syntheticity of all mathematical theorems is due to the syntheticity of mathematical axioms.

Here the roots of Kant’s thoughts in Euclid come in especially handy. It is almost embarrassingly clear what he means when we recall the structure of an Euclidean proposition. What misled earlier philosophers, Kant says, is that mathematical *inferences* (*Schlüsse*) proceed in accordance with the principle of contradiction, i.e., analytically. Kant does not say mathematical *proofs*. Now what is the part of an Euclidean proposition in which *inferences* are drawn? The proof proper, the *apodeixis*, of course. Hence, what Kant is saying is that his predecessors have correctly realized that the *apodeixis* is

analytical and from that have mistakenly concluded that the whole proposition can be established analytically. That this is his meaning is neatly verified by Kant's rider on the former of these two theses. The analyticity of all mathematical inferences is what "the nature of all *apodictic* [my italics] certainty requires." Kant is thus even verbally conforming to my reading. Moreover, from the first couple of lines of B17 it is seen that "the nature of all apodeictic certainty" did not, according to Kant, require that the whole argument for a proposition (or the proposition itself) be analytic. Hence, the requirement scarcely meant anything else but that the *apodeixis* part of a proof must proceed "in accordance with the principle of contradiction." This is supported by the remarkable contrast in the quoted passage between "mathematical inferences" and ways of coming to know the truth of basic mathematical propositions. This contrast alone should give pause to interpreters of the Cassirer-Brittan type.

Hence, what we find in the quoted passage is not a counterexample to my view but further evidence for it. It serves to confirm further that Kant located the analytical element of a geometrical proof in the *apodeixis* and the synthetic element in the *auxiliary construction* or *kataskeue*.

What I have said is connected with an exegetical and even textual problem. I have in effect been looking at what Kant says in the sentences I have quoted from B14 through the glasses offered to us by what he says in B17 (beginning with the words "Was uns hier" and ending with "hinzukommen muss, anhänge"):

What here causes us commonly to believe that the predicate of such apodeictic judgments is already contained in our concept, and that the judgment is therefore analytic, is merely the ambiguous character of the terms used. We are required to join in thought a certain predicate to a given concept, and this necessity is inherent in the concepts themselves. But the question is not what we *ought* to join in thought to a given concept, but what we *actually* think in it, even if only obscurely; and it is then manifest that, while the predicate is indeed attached necessarily to the concept, it is so in virtue of our intuition which must be added to the concept, not as thought in the concept itself.

This passage strongly supports my interpretation. The necessity of a synthetic proposition a priori is said to reside in the concepts involved and not to result from some special faculty called "intuition." The reason we do not actually think of the conclusion

(predicate) when we think of the premise (subject) is that the necessary connection between them can be seen only by means of an intuition. By “intuition” Kant does not here mean a faculty or any other source of knowledge, for if it were that, the necessity of the connection could not be inherent in the concepts themselves. What he obviously means is precisely the use of “intuitions” (particular representatives of general concepts) which takes place in the *ekthesis* and in the auxiliary construction of a proposition in Euclid. And this use of intuitions (instantiations) explicitly excludes all appeals to outside sources of information like geometrical imagination (“intuition”). Indeed, Kant repeatedly rules out such appeals. Hence the quoted passage from B17 supports my interpretation.

There is an intriguing possibility that the latter passage was indeed intended by Kant as a comment on the former. The B17 passage is obviously out of place; in its present location it cannot be read as a meaningful comment on the immediately preceding passage. Vaihinger has suggested that it should be attached to the preceding paragraph (“Ebensowenig . . . möglich ist”),¹¹ but this would break the unity of Kant’s discussion of the syntheticity of the *Grundsätze* of geometry. The apparently displaced comment is general; it pertains specifically neither to geometry nor to arithmetic (which Kant discusses in B15–16). Hence it follows naturally the initial paragraph of B14 which we have been discussing.

There is some evidence for this in the text. For instance, the reference to “apodictic judgments” (*apodiktische Urteile*) in B17 is naturally taken to pick up the reference to “apodictic certainty” (*apodiktische Gewissheit*) in B14. In the B17 passage Kant discusses certain unnamed but previously mentioned judgments (“*solche Urteile*”). The closest earlier mention of judgments (as distinguished from the *Grundsätze* which Kant discusses in the preceding three paragraphs) is in B14. Moreover, in the B17 passage Kant is discussing the reasons for certain unspecified mistakes, viz., mistaken classifications of some judgments or others as analytic rather than synthetic. Now the closest mention of such mistakes (*any* mistakes) in Kant’s text is precisely in the originally quoted passage.

Furthermore, if the passage in B17 were attached to the second paragraph of B16, as Vaihinger suggests, it would be repetitious: the need of intuition would be asserted twice over. In the second paragraph (the famous $7 + 5 = 12$ discussion) of B15, Kant mentions repeatedly what is actually *thought of* (*gedacht*) at different stages in the process of establishing this equation. These references

presuppose the criteria of syntheticity given by Kant in B17, according to which the real question is not what we have to think in connection with certain concepts but what we actually think in them. All this suggests that the B17 passage really belongs immediately after the B14 passage.

In any case, whatever the purely textual situation is or may be, interpretationally it has to be taken to pertain primarily to the passage from B14 that we have been considering. If so, my interpretation receives strong support.

The way of summing up my interpretation is therefore to compare what we have found about B14–17 by comparing it with what Kant says in Bxi–xii. It is instructive here to set parts of the two passages side by side (with transpositions):

B14, 17	Bxi–xii
<p>But the question [in attributing analyticity] is not what we <i>ought</i> to join in thought to the given concept but what we <i>actually</i> think in it. . . . Intuitions, therefore, must be called in.</p>	<p>The true method [of geometry is] not to inspect what [one] discerned in the figure, or in the bare concept of it, and from this, as it were, read off its properties</p>

B14, 17	Bxi–xii
<p>We are required to join in thought a certain predicate to a given concept, and this necessity is inherent in the concepts themselves.</p>	<p>. . . but to bring out what was necessarily implied in the concepts that he had himself formed a priori. . . .</p>
<p>It is manifest that, while the predicate is indeed necessarily attached to the concept, it is so in virtue of an intuition which must be added to the concept.</p>	<p>. . . and had put into the figure by which he presented it to himself.</p>
<p>[As] the nature of all apodictic certainty requires</p>	<p>If he is to know anything with a priori certainty, he must ascribe nothing to the figure</p>
<p>. . . all mathematical inferences must proceed in accordance with the principle of contradiction</p>	<p>. . . save what necessarily follows from what he has himself set into it in accordance with his concept.</p>

This comparison shows unequivocally that Kant located the intuitive and synthetic elements in geometrical and other mathematical arguments within the explicit proofs themselves, not in their premises, not in some nondiscursive appeal to our

geometrical (or temporal) intuition. In fact, such appeals are ruled out in so many words in Bxii.

The mistake of the line of thought I am criticizing may be partly based on another error. Because scholars like Cassirer believe that by the use of intuitions Kant meant appeal to our geometrical imagination, they cannot locate the intuitive and constructive element in geometry within an actual argumentative structure of, say, Euclidean propositions. As soon as we clearly see that by the use of intuitions in geometry Kant meant nothing more or less than the use of instantiation, such as those employed in Euclidean *ekthesis* and auxiliary construction, we can happily locate the intuitive and synthetic element right in the middle of axiomatic and deductive arguments and need not pursue the rainbow of syntheticity back to the axioms.

Even if Kant had departed from the spirit of his own basic ideas and mistakenly thought that somehow the syntheticity of mathematical truths can be traced back to that of the axioms and postulates, even so my interpretation would not be invalidated. For the all-important question remains *how* precisely the synthetic axioms and postulates enter into the proof of a mathematical theorem. As every historian of geometry knows, so-called postulates entered into geometrical arguments in Euclid as justifying the auxiliary constructions needed in most geometrical proofs. If so, the dependence of the syntheticity of a geometrical theorem on the status of the so-called postulates is compatible with my thesis. The syntheticity of a geometrical theorem is on my account recognized from the use of auxiliary constructions in its proof. Now we might equally well have said that it is recognized from the need of using so-called postulates as premises in its proof. There is hence no incompatibility whatsoever between tracing the syntheticity of a mathematical theorem back to its premises among the axioms and postulates of the branch of mathematics in question and saying that it is the use of auxiliary individuals in its proof that makes it synthetic.

Moreover, even if the likes of Cassirer and Brittan are correct and Kant thinks that the syntheticity of mathematical truths is due simply and solely to the syntheticity of those mathematical axioms from which they can be analytically derived, even then they face a formidable exegetical task which none of them has ever shouldered. If we do not lift the quoted passage from context, we find a truly remarkable thing. Instead of considering how the axioms of mathematics are intuited, Kant actually discusses in most of the

section in question the actual arguments by means of which sundry particular truths of arithmetic and geometry can be established. What the rationale of this procedure—this sudden change of logic—could conceivably be has never been explained by the adherents of the Cassirer-Brittan interpretation.

All told, my interpretation thus receives strong support both from the examination of Kant's historical background and from the analysis of particular Kantian texts—and especially from both of them combined.

NOTES

1. Most of my work on Kant has been collected in my *Logic, Language-Games, and Information: Kantian Themes in the Philosophy of Logic*, (Oxford: Clarendon Press, 1973, chaps. 5–10; or in my *Knowledge and the Known*) Dordrecht: D. Reidel, 1974, chaps. 6–10. See also my papers "On Kant's Notion of Intuition (*Anschauung*)" in Terence Penelhum and J. J. MacIntosh, eds., *The First Critique: Reflections on Kant's "Critique of Pure Reason"* (Belmont, Calif.: Wadsworth, 1969), pp. 38–53; and "Kantian Intuitions," *Inquiry* 15 (1972): 341–45.

2. See Charles Parsons, "Kant's Philosophy of Arithmetic," in Sidney Morgenbesser et al., ed., *Philosophy, Science, and Method: Essays in Honor of Ernest Nagel* (New York: St. Martin's Press, 1969), pp. 568–94; Manley Thompson, "Singular Terms and Intuitions in Kant's Epistemology," *Review of Metaphysics* 26 (1972): 314–43; and Gordon G. Brittan, Jr., *Kant's Theory of Science*, (Princeton, N.J.: Princeton University Press, 1978), pp. 49–56. All these criticisms are marred by total mistakes. As I pointed out in "Kantian Intuitions" (note 1 above), Parsons misunderstands the force of the Kantian concept of intuition. Thompson likewise misunderstands completely the idea that Kantian intuitions have an immediate relation to their objects. He writes: "To refer immediately, then [*sic*], is to refer to an object by means of marks or characteristics that it alone possesses." This is diametrically opposed to what Kant intends. For him the immediacy of intuition representation lies precisely in that it does *not* take place by means of "marks or characteristics." This is brought out clearly in Kant's definition of intuition in A320/B376–77 and in his *Logic*. Brittan misunderstands Kant's comments on his predecessors in B14, as I shall argue below.

3. For details and for evidence see the literature referred to in note 1 above.

4. See F. Kaulbach, entry on "Anschauung" in Joachim Ritter, ed., *Historisches Wörterbuch der Philosophie* (Basel, Schwabe & Co., 1971), 1:340–47; entry on "Anschauung" in Jürgen Mittelstrass, ed., *Enzyklopedie der Philosophie und Wissenschaftstheorie*, (Mannheim: Bibliographisches Institut, 1980) 1:120–21.

5. At attempt to do so is made in Jaakko Hintikka et al., *Aristotle on Modality and Determinism*, North Holland, Amsterdam, 1976.

6. Wolff's influence tended to strengthen, not to weaken, Euclid's dominance in this respect.

7. A philosophical reader will find a statement of this procedure in Leibniz's *Nouveaux Essais*, book 4, chap. 17, sec. 3.

8. See my "C. S. Peirce's 'First Real Discovery' and Its Contemporary Relevance," *Monist* 63 (1980): pp. 304–15.

9. The second sentence of *La Geometrie* spells out this analogy explicitly: "Just as arithmetic consists of only four or five operations, namely, addition, subtraction,

multiplication, division, and the extraction of roots . . . , so in geometry, to find required lines it is merely necessary to add or to subtract other lines; or else, taking one line which I shall call unity in order to relate it as closely as possible to numbers, and which can in general be chosen arbitrarily, and having given two other lines, to find a fourth line which shall be to one of the given lines as the other is to unity (which is the same as multiplication); or, again, to find a fourth line which is to one of the given lines as unity is to other (which is equivalent to division); or, finally, to find one, two, or several mean proportionals between unity and some other line (which is the same as extracting the square root, cube root, etc., of the given line. And I shall not hesitate to introduce these arithmetical terms into geometry, for the sake of greater clarity.”

10. Ernst Cassirer, “Kant und die moderne Mathematik,” *Kant-Studien* 12 (1907): 1–40; Brittan, *Kant’s Theory of Science*, pp. 55–58.

11. H. Vaihinger, *Commentar zu Kants Kritik der reinen Vernunft* (Stuttgart: W. Spemann) 1881, 1:303–304.