***Leibniz: De quadratura arithmetica circuli ellipseos et hyperbolae cujus corollarium est trigonometria sine tabulis*. Critical edition and commentary by Eberhard Knobloch. Göttingen: Vandenhoeck and Ruprecht, 1993, 160pp.**

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On numerous occasions Leibniz stressed the importance of providing the new infinitesimal calculus with the solid foundaiions it required by means of rigorous proofs. His treatise on the arithmetical quadrature of the circle, the ellipse and the hyperbola shows clearly that this was in fact a foremost consideration right from the outset.

The original manuscript was left behind by Leibniz on his departure from Paris in 1676 in the hope that its publication would lead to his being accepted as member by the Académie Royale des Sciences. Through unfortunate circumstances the project fell through and while being conveyed to Hanover in 1680 the manuscript went missing. What in all probability is a fair copy of the original prepared in Paris by his friend Soudry later found its way into Leibniz' papers in Hanover. Although part of the work was published in 1934 by Lucie Scholtz,[1] only now do we have a critical edition of the whole manuscript prepared meticulously to the standard of the *Academy Edition* by Eberhard Knobloch.

The treatise, in which the infinitesimal geometry known in 1676 is set forth, provides a uniform basis for higher analysis using indirect proofs and through considerations on limits. Already in 1673 Leibniz was acquainted with arithmetical quadratures based on infinite series through the work of Brouncker and Mercator. Shortly thereafter he was able to discover independently the arithmetical quadrature of the circle based on the alternative convergent series which bears his name (p. 79) and to develop from this a general method applicable to all conic sections. The root of this method is the transmutation theorem, permitting one with the help of an auxiliary curve to transform conic sections into rational figures, a rigorous proof of which is provided in propositions 6 and 7 of the treatise. This theorem along with the theorem of cycloidal segments (proposition 13) and the harmonic triangle (propositions 39 and 40) counts among the most important results achieved by Leibniz during his stay in Paris.

Leibniz describes proposition 6 as "most thorny" (*spinosissima*); in it he seeks to show in an excessively careful manner how a rectilinear stepped space or a polygon in continuing development can be brought to differ from each other and from a curve by an amount that is smaller than any quantity that can be gıven In doing so he lays the foundation for the method of indivisibles "in the most reliable way," supplying a proof that the areas of figures can be calculated by means of the "sums of lines" (p. 29). Like Pascal and Roberval, Leibniz interprets these lines or indivisibles as rectangles with equal breadths of what he calls "indefinite smallness" (*indefinitae parvitatis*) (p. 39).

Leibniz emphasizes more than once that the method of indivisibles can lead to error when not used in this, its mature form (pp. 69, 133). His favourite example is the paradox of the hyperbola: if one sums up in pairs equally large areas between two ordinates or the corresponding abscissae, a contradiction with the (in Leibniz's view) universally valid Euclidean axiom that the whole is greater than the part is reached (p. 67). Nevertheless, it is misleading when he uses the term "indefinite smallness," as this suggests agreement with authors like Hobbes and Cavalieri who sought to avoid the use of the infinite in mathematics. In fact Leibniz has in mind the concept "as small as one likes" (*utcunque parvus*), which, along with the concept "smaller than any quantity that can be given" (*minus quavis assignabili quantitate*), he uses in order to define the infinitely small.

A detailed discussion on the concept of infinity is to be found in the first version of the scholium to proposition 11 of the treatise (p. 132f.). Referring to proofs for the finite area or volume of certain infinitely long geometrical spaces by authors like Torricelli and Grégoire de Saint-Vincent, including the former's work on the "acute hyperbolic solid," Leibniz remarks that the results are not quite so remarkable as first meets the eye, since the mental operation enabling the infinite space to be measured rests on a fiction, i.e. on a line which is assumed to be terminated, but which in fact is infinite. This corresponds to the concept of the infinite and the infinitely small which on Leibniz's view can be used in calculation: the infinite is larger than any quantity that can be given, while the infinitely small is likewise smaller than any quantity that can be given. Such infinite quantities are not in a true sense infinite, but rather are useful fictions making possible results which are demonstrably correct. In contrast, neither points or minima nor the unlimited or maxima can be the object of mathematical considerations. The question of existence, which Leibniz regards as being the task of the metaphysician, is hereby avoided (cf. p. 69). The geometrician is content to prove that which follows from what he assumes.

At the same time Leibniz recommends caution in working with the infinite. Calculating with it can be slippery, he says, without the guiding principle of a proof (p. 67). In particular absurdities like the paradox of the hyperbola can arise when the infinitely small is identified with a point or with zero. As he puts it in the context of reaching the asymptote, we cannot always leap from the property of a finite abscissa to the property of an infinite space (p. 67). However, through the use of indirect proofs, ensuring that the error is smaller than any quantity that can be given, Leibniz believes that his method of quadrature based on fictive quantities is not only exact, but also that it represents a substantial simplification when compared to the classical method of exhaustion, not least because it does not require both the inscription and circumscription of the curvilinear figure whose area is sought (p. 35). Uniformity and simplicity of method is indeed for Leibniz an overriding concern (cf. p. 80). The detailed treatment of individual problems as well as the rigorous proofs, while putting his method on a sound base, can, as he points out, nevertheless be ignored without danger by future geometricians when they come across similar reasoning (p. 41).

But even the most solid reasoning may be superseded in time. When Jean Bernoulli encouraged Leibniz to finally publish his treatise years later, the author replied that it might have been well received at the time of writing, but that now it would be more likely "to please beginners in our method than you" (GM III 537).

In this respect Leibniz was probably correct, though a number of results, including the geometrical quadrature of the logarithmic curve (proposition 46), would still have represented a contribution to contemporary mathematical knowledge. Nevertheless, as Knobloch has outlined in a number of recent articles,[2] the treatise is of considerable importance not only in respect of the development of Leibniz's mathematical ideas during his sojourn in Paris, but also because of the insight it provides into his understanding of the infinite. Not least for this reason the edition of *De quadratura arithmetica circuli* deserves attention not only from historians of mathematics, but also and in particular from those interested in seventeenth century discussions on the problem of infinity.

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[1]L. Scholtz, *Die exakte Grundlegung der Infinitesimalrechnung bei Leibniz* (Teildruck), Marburg, 1934.

[2]E. Knobloch, "Leibniz et son manuscrit inédité sur la quadrature des sections coniques," in: *The Leibniz Renaissance*, International Workshop (Firenze, 2-5 giugno 1986), ed. Centro Fiorentino di Storia e Filosofia della Scienza, Florence 1989, pp. 127-151; "Progrès et tâches futures de la recherche leibnizienne en mathématiques," in: *Les Etudes philosophiques* 1989, pp. 161-170; "L'infini dans les mathématiques de Leibniz," in: *L'infinito in Leibniz. Problemi e terminologia.* Simposio internazionale del Lessico Intellettuale Europeo e della Gottfried-Wilhelm-Leibniz-Gesellschaft, Roma, 6-8 novembre 1986, ed. A. Lamarra, Rome 1990. pp. 33-51; "Les courbes analytiques simples chez Leibniz," in: *Sciences et techniques en perspective* 26, 1993, pp. 74-96