(LH XXXV, I, 14, bl. 23-24)

Uniformis locus seu **sibi congruus**, dici potest, cujus partes congruenter terminatae sunt congruentes.

At **locus sibi similis** est, cujus partes similiter terminatae sunt similes. Loca sibi similia sunt non alia quàm recta, planum, spatium ipsum. Loca uniformia sunt omnia loca sibi similia, et praeterea alia, nempe ex lineis quidem arcus circuli et helix cylindrica, ex superficies verò sphaerica et cylindrica. Nam hos locos non esse sibi similes. Exemplo arcus circuli intelligi potest, cujus duae partes, id est duo arcus ejusdem circuli, similiter sunt terminatae, per duo scilicet extrema puncta, non tamen sunt similes inter se, cum eandem rationem ad rectas extrema jungentes non habeant, adeoque discerni possint; quemadmodum et discerni possunt angulo quem captant aliisque modis.

Uniformis locus…

A **locus** can be called **uniform** or **self-congruent** if its congruently bounded parts are congruent.

On the other hand, a **locus** is **self-similar** if its similarly bounded parts are similar.[5] The only self-similar loci are the straight line, the plane, and space itself. Uniform loci include all self-similar loci and, besides, others—that is to say, among the lines, the arc of a circle and the cylindrical helix and, among the surfaces, the spherical and the cylindrical ones.[6] In fact, these loci are not self-similar. For example, an arc of a circle can be given, in such a way that two parts of it, i.e. two arcs of the very same circle, are similarly bounded by two extreme points, and yet these parts are not similar to one another, because they do not have one and the same ratio with respect to the straight lines joining the extreme points, and therefore they can be distinguished. In fact they are distinguishable by the angle which they make, and also in other ways.

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[5] The determination of a figure through its boundary (*ambitus*) under an equivalence relation is a fundamental feature of Leibniz's late geometry. Cf. the *Initia rerum mathematicarum metaphysica*, GM VII, pp. 21 and 27.

[6] The importance of the circular helix in foundational studies was well known and already mentioned by Geminus (a full discussion of it can be found in PROCLUS DIADOCHUS, *In Primum Euclidis*, edited by G. Friedlein, Leipzig 1873, pp. 104-106). It presented the problem that a solid line joins the straight line and the circle in making up the simplest curves, called ὁμοιομερεῖς, i.e. similar in any part of them (though the reference is here to a property of congruence, not of similitude). In any case, very few geometers ever regarded it as having the same relevance as the other two lines. Homoeomeric curves were also discussed in Leibniz's times, as of course Proclus's commentary was very well known, but referred to in the vaguest terms. See, for example, G. Galilei, *Dialogo sopra i massimi sistemi*, Prima Giornata, in *Le Opere*, Firenze 1968, vol. 7, p. 40; and T. HOBBES, *De Corpore*, XIV, 3, in *Opera Philosophica quae latine scripsit omnia*, edited by W. Molesworth, London 1839-45, vol. 1, p. 156. The worth of Leibniz's way of discussing it lies therefore in his starting from much sounder premises (for instance, in the clear distinction he makes between congruence and similitude, two terms Galileo does not use technically), which also allows him to reach much further in the determination of minimum problems (cylindrical geodesic curves). The problem may have been first hinted at in a 1679 study on the *via minima* (*La caractéristique géométrique*, p. 272), but it really comes to the fore in Leibniz's mature years, and is mentioned again in the *In Euclidis* πρῶτα (GM V, p. 199).

Sed recta omnium locorum maximè sibi simile est, nam cùm omnes rectae partes similiter sunt terminatae, per duo scilicet puncta extrema, hinc omnes rectae partes sunt inter se similes, adeoque et similes toti. Nec alius datur in natura rerum locus praeter rectam in quo pars quaevis sit similis cuivis. Planum quidem et spatium seu corpus mathematicum sibi intus similia sunt, id est si terminus non respicias partes eorum discernere non potes, ut qui in ingenti planitia oberrat, aut in media maris abysso (entre deux eaux) divagatur, ut nec fundum nec superficiem percipiat, ubi sit cognoscere nequit; sed si terminus superficiei planae aut fundum vel superficiem aquae attingere ipsi detur, fieri potest, ut illic notet discrimen; nam planum variis lineis, et spatium variis superficiebus terminari potest. Si tamen terminatio plani sit similis terminationi alterius plani jam rursus ratio discernendi amissa est, unde circulus à circulo, polygonum regulare à polygono regulari totidem laterum, discerni non potest, nec intus nec extra. Idemque est in spatiis solidis pluribus similiter terminatis, nam sphaerae duae, & cubi duo sunt similes inter se tam intus quam extra. Et haec causa est cur locum sibi similem definierim: cujus partes similiter terminatae sunt similes; ut scilicet non tantum rectam comprehenderem, perfectissimè similem sibi, sed etiam planum et spatium, quae tantùm intus sibi similia sunt.

Habemus ergo tres gradus: **Locum sibi simillimum**, in quo partes omnes sunt similes, et hoc proprium est rectae soli; | **Locum intus sibi similem** cujus partes similiter terminatae sunt similes, qualis est (praeter rectam) planum et spatium; et denique **Locum uniformem** cujus partes similiter terminatae aequales (seu congruenter terminatae) sunt similes, adeoque et congruentes; seu cujus partes

On the contrary, a straight line is the most self-similar of all loci. In fact, as all parts of a straight line are similarly bounded through two extreme points, so all parts of the straight line are similar between them, and therefore similar to the whole. Nor in nature can any locus be given other than a straight line, in which any part would be similar to any other.[7] A plane, however, and space, or a mathematical body, are similar inside, i. e. you cannot distinguish their parts unless you look at their boundaries. In much the same way, if a man wanders through a vast plain, or happens to go adrift in the deepest regions of the sea (*entre deux eaux*) and see neither the bottom nor the surface of it, he will not even know where he is; but if the border of the plane surface can be seen or the bottom or the surface of the water can be reached, then he will be able to notice a difference—in fact, a plane can be bounded by various lines, and so can space by various surfaces. If however the boundary of a plane is similar to the boundary of another plane, then the criterion for distinguishing between them is lost once again, so that one cannot discriminate – either inside or outside – a circle from another circle, nor a regular polygon from another one having as many sides. This is also the case with many solid spaces having similar boundaries. For instance, two spheres and two cubes are similar to each other both inside and outside. And this is why I would define a self-similar locus as the locus whose similarly bounded parts are similar—in order to include in such a definition not only a straight line, which is perfectly self-similar, but also a plane and space, which are only self-similar inside.

As a result, we will have three degrees: **a most self-similar locus**, in which any part is similar to any other, which is a property of the straight line alone; **an inside self-similar locus**, whose similarly bounded parts are similar, such as (besides the straight line) a plane and space; and finally, **a uniform locus**, whose

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[7] Note that the definition of a straight line through self-similarity is only sound as long as by straight line we mean the *recta terminata* (a bounded rectilinear segment), as Leibniz says in some other writings. The uniqueness of the straight line among the self-similar loci is stressed again and again in the late years. See for example *In Euclidis πρῶτα*, GM V, p. 185. In a text Leibniz wrote in the same year, actually, he also noted that such a definition of a straight line could very well apply to a point—because a portion of a point is the point itself, and because any point is similar to itself thanks to the reflexive property of similitude (this still unpublished text is in LH XXXV, I, 13, bl. 12-13). The definition of a straight line as a self-similar curve was hinted at in one of the "*théorèmes connus naturellement*" in Pascal's introduction to geometry (copied by Leibniz in 1676; see J. ITARD, *«L'introduction à la géométrie» de Pascal*, in *L'oeuvre scientifique de Pascal*, Paris 1964, p. 107).

congruenter terminatae sunt congruentes; talis praeter rectam, planum et spatium, est arcus circularis, helix cylindrica, superficies sphaerae, superficies cylindrica. Habemus ergo locum sibi simillimum, cujus omnes partes sunt similes; locum intus sibi similem, cujus partes similiter terminatae sunt similes; locum uniformem cujus partes congruenter terminatae sunt congruentes. In omni loco uniformi interminato omne punctum similiter situm est, ut in recta, circuli arcu, helice cylindrica, plano, superficie sphaerica, superficie cylindrica; seu in loco uniformi nisi terminus respicias punctum à puncto discerni nequit. Itaque **Locum uniformem** etiam sic definire licebit, ut sit ille cujus punctum quodvis intus similiter situm est. Intus inquam id est sinè respectu terminorum. Omnis autem locus uniformis hoc habetur, ut pars ejus super toto (quantum satis est producto) incedere possit; vel etiam pars una super alia.

Ex his illustratur similitudinis et congruentiae natura. Et quidem congruentia jam usi sunt geometrae, at similitudine non aequè, quia ejus definitionem non habebant. Quam ego demum generalem dedi, non ab angulis sumtam, quae casum tantum praebent similitudinis, sed à principio altiore, nempe à ratione discernendi. Notavi etiam ut similitudinem & congruentiam, ita quoque homogeneum & aequale discriminavi. Nam Homogenea sunt loca quae transformari possunt in similia; sed aequalia sunt quae transformari possunt in congruentia. |

Geometrae postularunt rectae et plani descriptionem, modum verò exhibendi non dedere, nam cum regulam in describenda recta adhibent, jam rectam in regulae ipsius centro descriptam sumunt. Duo autem modi sunt definiendi rectam, unus per puncta, alter per motum continuum; ex his prior magis est geometricus, posterior magis organicus, quamquam ita geometriam rectè accipere solamus, ut similarly and equally bounded (or whose congruently bounded) parts are similar and also congruent; such is the case with – besides the straight line, plane, and space – the circular arc, the cylindrical helix, the spherical surface, and the cylindrical surface. Thus, what we have here is: a most self-similar locus, all parts of which are similar; an inside self-similar locus, whose similarly bounded parts are similar; and a uniform locus, whose congruently bounded parts are congruent. In every unbounded uniform locus any point is similarly situated, just as in a straight line, an arc of a circle, a cylindrical helix, a plane, a spherical surface, a cylindrical surface; that is, in a uniform locus, unless you can see its boundaries, no point can be distinguished from any other point. Hence, a **uniform locus** may also be defined as a locus in which any inside point whatsoever is similarly situated. Inside, I say, that is without taking its boundaries into consideration. Every uniform locus has this property, that any of its parts – as long as it is increased enough – can overlap the whole; and so can any part overlap any other.

The above elucidates the nature of similarity and congruence. Indeed, while geometers have been familiar with congruence, they are not as familiar with similarity, because they lack its very definition. The definition I have finally given is a general one, and it has not been deduced from the consideration of angles, which represents only one instance of similarity, but from a deeper principle, that is from the principle of discerning.[8] As I have discussed similarity and congruence, I have also distinguished between homogeneity and equality. In fact, the loci that can be transformed into similar ones are homogeneous; while the loci that can be transformed into congruent ones are equal.[9]

Geometers have postulated the description of a straight line and a plane, but they have not given any way of showing it; in fact, when they employ their rule for describing a straight line, they already assume the described straight line in the core of the rule itself. There are however two ways of defining a straight line, one through points and the other one through continuous motion; the former one

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[8] Leibniz arrived at the perceptual definition of similarity in 1677 (Leibniz to Gallois; GM I, p. 180; A, III, 2, n. 79, p. 227-28; A II, 1, n. 158, p. 380) and treasured it as one of the best results of his mathematical and philosophical investigation; in his late years its importance grew enormously. In the mid-nineties Leibniz repeatedly wrote that the Euclidean definition of similarity was just a particular case of his more general phenomenological definition, and sometimes he even tried to deduce from it the formal mathematical one. See *De Analysi Situs*, GM V, pp. 179 and 181-82; *Specimen Geometriae Luciferae*, GM VII, pp. 281-82.

[9] The amplest essay on this issue is to be found in the *Specimen Geometriae Luciferae*: «Porro eodem fere modo quo ex congruis nascuntur aequalia, etiam ex similibus nascuntur Homogenea...» (GM VII, p. 282).

et organicam seu phorographiam comprehendat. Et quidem per motum exhibetur recta, si assumatur corpus (non excavatum) et duobus punctis immotis moveatur, id est gyratur, tunc locus omnium punctorum quiescentium erit recta; locus autem successivus puncti mobilis erit arcus circuli. Per puncta autem ut exhibeamus rectam, prius exhibeamus planum; est autem planum locus omnium punctorum similiter se habentium ad duo puncta data. Nempe si sit punctum X, eodem modo se habens ad A et ad B, tunc locus omnium X erit planum $\overline{X}$. Itaque sic puncta plani quotlibet inveniri possunt, si circa A et B, intelligantur duae superficies sphaericae inter se aequales, et sibi occurrentes, puncta ambabus superficiebus communia, erunt in plano $\overline{X}$ quod eodem modo se habet ad A quo ad B. Et si aliae superficies sphaericae aequales prioribus utcunque majores aut minores sibi occurrentes assumantur puncta concursum rursus erunt in eodem plano $\overline{X}$.

Ita modum habemus plani puncta quotcunque determinandi.

Porro rectae puncta quotcunque eodem modo determinabimus in ipso plano. Nam sumtis in plano aliquo duobus punctis C et D, sumatur punctum Y eodem modo se habens ad C et ad D, et locus omnium Y erit recta $\overline{Y}$.

Itaque sic puncta rectae quotlibet inveniri possunt: si in plano circa C et D intelligantur duae circumferentiae circuli descriptae inter se aequales, et sibi occurrentes, puncta ambabus communia, cadent in rectam eodem modo se ad C habentem quo se habet ad D. Et hae rectae planique definitiones aptissimae sunt | ad Constructionem Geometricam; neque enim ad eam aliud postulatur, quàm extensum moveri posse uno puncto manente immoto; hoc est in spatio posse describi superficiem sphaericam, in plano lineam circularem, circa punctum is more geometrical, the latter is more organic, although we are accustomed to regard geometry as also including *organica* i.e. phorography. Indeed a straight line can be shown through motion—provided a (non-hollow) body is given which moves around two fixed points, i.e. which rotates, then a straight line will be the locus of all motionless points; the locus produced by a point in motion will be an arc of a circle.[10] In order to show a straight line through points, we will first show a plane; in fact, a plane is the locus of all points similarly situated with respect to two given points. In other words, let a point X be given, bearing the same situation with respect to A and B, then the locus of all points X will be a plane $\overline{X}$. Thus as many points of a plane as one wishes can be found: provided two spherical surfaces are taken that have A and B as their centers, are equal to one another, and meet one another, then the points common to both surfaces will lie in the plane $\overline{X}$ that bears the same situation with respect to A and B. And if other equal spherical surfaces, greater or smaller than the preceding ones, are taken that meet one another, once again their meeting points will lie in the same plane $\overline{X}$.[11]

Thus we possess a way to determine any number whatsoever of points of a plane.

Furthermore, in the very same way we will determine any number whatsoever of points of a straight line in the plane. In fact, let two points C and D be taken in a plane, and let a point Y also be taken that bears the same situation with respect to both C and D, then the locus of all Y will be the straight line $\overline{Y}$.

Thus in this way we can find any number of points of a straight line: in a plane, let two circumferences be described that have C and D as their centers, are equal to one another, and meet one another, then those points in common between them will fall into the straight line that bears the same situation with respect to both C and D. And these definitions of a straight line and a plane are very convenient for all Geometrical Construction; for nothing else is required than the possibility for an extensum to move while one point remains fixed; the possibil-

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[10] The definition of a straight line as a rotation axis goes back to the ancient times (it can be found in PROCLUS, *In Primum Euclidis*, p. 110, perhaps with reference to Heron), but it had a certain revival in the seventeenth century. Among the authors surely known to Leibniz, Roberval has made ample use of it (on the subject, see V. JULLIEN, *Les étendues géométriques et la ligne droite de Roberval*, in «Revue d'Histoire de Sciences», 46, 1993, pp. 493-521), as well as Giordano, who explicitly mentions it in his letter to Leibniz (GM I, p. 198; A III, 4, n. 217, p. 425). The first studies by Leibniz on this subject (in search for a real definition for the rotation axis), can be found in *La caractéristique géométrique*, p. 66.

[11] This definition is recursive in Leibniz writings on *analysis situs*.

datum; quae constructiones sunt omnium simplicissimae. Quamquam nec superficie sphaerica integra descripta opus sit, nec circuli arcu, sed sufficit duo extensa congruentia punctis A et B (vel C et D) applicari, et applicato puncto immoto ita moveri, ut punctis respondentibus sibi occurrant, et locus occursus erit punctum rectae vel plani quaesitum.

Si rectam hac methodo velimus describere non prius descripto plano, id quoque praestari potest ex eo, quod recta est locus omnium punctorum ad tria puncta data eodem se modo habentium. Possibile non est ut detur punctum eodem modo se habens ad quatuor puncta in eodem plano non posita, seu a quatuor punctis non ejusdem plani aequidistans.

Quoniam primitiva rectae definitio est, ut sit linea sibi similis operae pretium erit ostendere, locum punctorum a tribus punctis aequidistantem esse sibi similem. Sed et aliter ostendi potest talem locum esse rectam. Nam ostendimus rectam esse locum monologorum ad puncta duo. Itaque moto extenso duobus punctis immotis quiescere rectam, adeoque eodem modo se habere ad totum locum successivum puncti moti, seu ad circumferentiam circuli quam describet, adeoque et ad tria puncta eam circumferentiam determinantia. Est igitur recta talis locus seu recta est locus punctorum ad tria puncta tautologorum sed vicissim omnia talia puncta cadunt in rectam, quia durante motu quiescunt.

Dari Lineam unicorum ad duo puncta seu monologorum in plano ita ostendi potest immediatè (non hadibita definitione rectae per similitudinem): sit ad puncta A, B, punctum C, id aut est unicum ad A, B, aut geminum; si unicum jam habetur quaesitum; sin sit geminum, adeoque aliud sit ei respondens D; accedant sibi ambo similiter seu aequali velocitate; erit punctum occursus E, suae ad A et B ity, in other words, to describe a spherical surface in space and a circular line in a plane, about a given point; which constructions are the simplest of all. In any case, there is no need even to describe the whole spherical surface or the whole arc of a circle, but all that is needed is that two congruent extensa be fixed in points A and B (or C and D), and, provided a point has been fixed, the extensa move in such a way that any corresponding points meet; then the locus in which they meet will be the point of the straight line or plane we wanted to determine.

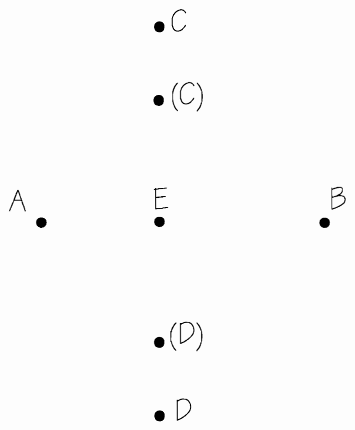
If we want to draw a straight line using this method without having first described a plane, that can also be done, for a straight line is the locus of all points that bear the same situation with respect to three given points. There exists however no point that bears the same situation with respect to four points not lying in the same plane, that is to say there exists no point equidistant from four non-coplanar points.

As the primitive definition of a straight line is the self-similar line, it is worth showing that a locus equidistant from three points is self-similar. However, there is another possible way of showing that such a locus is a straight line.[12] In fact, I have shown that a straight line is the locus of all the points that are monologous to two given points. Therefore, provided an extensum is set into motion about two fixed points, the straight line will remain motionless, and therefore it will bear the same situation with respect to the whole locus described by the point in motion, that is with respect to the circumference described by it, and all the more so with respect to three points determining the circumference itself. Thus, a straight line will be such a locus, i.e., a straight line is the locus of all the points that are tautologous to three given points but, conversely, all the points that remain motionless during the motion fall into the straight line.

That we can have a line made up of points unique by situation to two other points, i.e. that we can have a line of monologous points in a plane, can also be immediately shown (without resorting to the definition of a straight line through similarity) as follows: let a point C be given bearing one and the same situation with respect to points A, B, then it will either be unique to points A and B, or it will be one of a pair; if it is unique, then we already have what we were looking for; if on the contrary it is one of a pair, then there will be another point D corresponding to it; let C and D get close to one another in a similar way, that is at the

[12] In fact, the demonstration of the equivalence between the definition by self-similarity and that by equidistance is very arduous. Thus Leibniz in this text (and in many others too) seems to give up, and he only demonstrates the equivalence between the definition of the straight line by equidistance and the definition by determination. This is much simpler, in fact, for both definitions are grounded in one and the same situational relation—congruence.

relationis unicum; nam cum quaelibet alia loca ut (C), (D) sint gemina, et non nisi gemina, ita ut non detur aliud punctum in eodem plano, praeter (C) et (D) eodem modo se habens ut ipsa ad A et B, et in E gemina illa coincidant, utique non dabitur ibi aliud quàm unicum. Idem demonstrari potest ex sectione duorum planorum, quia est absolutè sine respectu ad planum; ponamus omnia illa puncta ad se invicem accedere, congruenter, necesse est ea simul coincidere in uno puncto quod utique est unicum suae relationis, cum, semper omnia loca appropinquantium ut C et D, simul sumta sint sola talium, ergo denique et in ultimo ubi omnia fiunt unum.



same speed; then they will meet in a point E unique as to its relation to both A and B; in fact, since all other loci whatsoever, say (C) and (D), are in pairs, and only in pairs, so that there is no other point in the same plane, besides (C) and (D), that bears the same situation with respect to A and B, and since those in-pair points meet in E, then absolutely no point will be given other than a unique one. As it bears absolutely no relation to the plane, the same thing can be demonstrated through the intersection of two planes; suppose all those points get reciprocally closer to each other in a congruent way, then they all will necessarily end up by coinciding in one point that is unique as to its relation; because all the loci of the points that get reciprocally closer, just as C and D, taken together are always formed by points unique by relation, and therefore so are they in the end, when they all become one.[13]

[13] The last line of this sheet in the manuscript is barely decipherable as Leibniz was running out of space. (A better version of this demonstration can be found in § 26 of Leibniz's *Analysis geometrica propria* (1698; GM V, p. 176), written probably for Bodenhausen and representing a good point of arrival for the geometrical research of the 1690s.) In the margins, finally, Leibniz tries to sketch another demonstration of the same theorem which may read as follows: «Sic quoque ratiocinari licebit. Sint plura puncta eodem modo ut C se habentia ad A et B, quae vocantur X, sit punctum E eodem modo se habens ad $\overline{X}$ totum seu ad locum $\overline{X}$, unicum suae ad $\overline{X}$ relationis, id erit unicum ad A, B relationis, et potest esse † ad A et B. Tantum probandum tale E dari».