Gottfried Wilhelm Leibniz: On Estimating the Uncertain

Translated[1] by Wolfgang David Cirilo de Melo

A *game* is *fair* if there is the same proportion of hope to fear on either side.[2] In a fair game, the hope is worth as much as it has been bought for because it is fair that a thing should be bought for what it is worth, and the fear is as great as the price of the hope.

Axiom: If players do similar things in such a way that no distinction can be drawn between them, with the sole exception of the outcome, there is the same proportion of hope to fear.

This can be demonstrated by metaphysics: where the appearances[3] are the same, the same judgement can be formed about them, that is, the way of thinking about the future outcome is the same; and the thoughts about the future outcome are hope or fear.

If the pool is formed by common, equal contribution of the players, if each one is playing in the same way, and if for the same outcome the same prize or the same penalty is fixed, the game is fair.

Once the same conditions have been posited,[4] each one's hope is worth as much as he contributed.

For when making his contribution, he buys his hope with a price, and with a fair price at that because the game is fair (*by the preceding arguments*). So the hope is worth as much as he bought it for, that is, it has as much value as he contributed.

In general, once that the same conditions are posited the estimated value of a man's hope is the portion of the pool that he has, be it that an equal contribution has been made, be it that the pool has been provided from somewhere else.

For concerning the estimated value of the hope, it does not matter where the pool has come from (this pertains to the fear of losing, which is zero if the contribution is zero). So the hope will be worth as much as it was when the pool was formed by contributing to it. At that time, however, the estimated value of the hope was a man's portion of the pool (that is, as much as each one had contributed, thus he as well).

[1]This is a translation of: Leibniz, G. W. (1678). De incerti aestimatione. In *Gottfried Wilhelm Leibniz: Sämtliche Schriften und Briefe*, Vol 6, pp. 91–101. Akademie Verlag. Sechste Reihe, Vierter Band, Teil A.

[2]'Hope' and 'fear' signify 'chances' and 'risk'. I have chosen to keep the translation literal.

[3]Leibniz is going from indifference – the inability to distinguish between things – to equipossibility.

[4]Leibniz is emphasizing the identity of conditions for the players before the game starts.

To put it differently: let us suppose that the whole pool pertains to all and that everyone's hope is equal; if the players broke the game off and wanted to distribute the pool according to the hope or the claim to it, with the intention of profit, a man's share would be owed to each one.[5]

The more people there are gambling on the same pool in a fair game, the smaller the hope of winning is. The reason is that it is worth less, because the more players there are, the smaller a man's portion is.

But the fear of losing is also smaller, otherwise the game would not be fair.

The more players there are who contribute to the pool equally, the greater the hope for profit is, but the greater is also the fear of losing.

For with one coin I can gain three, but that does not mean that I am acting more prudently than someone who plays against a single person. Thus it is necessary that the fear of losing has also been increased.

Let it be understood that the pool is divided into as many parts as there are players, and that the parts are A, B, C etc. The estimated value of my hope will be $A + B + C$ etc divided by the number of parties or players.

The reason is that the pool is $A + B + C$, and the portion of it which a man has measures his hope.

It is the same even if the whole pool is not exhausted after the parts have been divided.

For it is clear that a man's portion of the remainder belongs to each one; so each one has a portion both of the whole and of course of $A + B + C$, which has been taken away.

If I expect either A or B with the same hope, the estimated value of my hope will be half the sum of A and B.

For it is exactly as if there were two players and A was awarded to one by the outcome, B to the other.

If I expect either A or B or C with the same hope, the estimated value of my hope will be a third of the sum of A, B and C.

The demonstration is the same as before.

[5] I regard *ludendi* as dependent on *iuris*. A player's *ius ludendi* is his claim to the pool or to parts of it.

To put it in a general way, if the various successful outcomes can be kept separate in an activity, the estimated value of my hope will be the sum of possible successes collected from all outcomes, divided by the number of outcomes.

And in the same way, if the various negative outcomes can be kept separate in a transaction, the estimated value of my fear will be the sum of possible losses collected from the various cases, divided by the number of outcomes or cases.

We can reach the same conclusion in this way:

Probability is the degree of possibility. *Hope* is the probability of having. *Fear* is the probability of losing. The *estimated value* of a thing is as high as each one's claim to it.

If several people share a thing, or if a thing is common to several people through the same claim, each man's claim is his share of the claim to the whole thing.

If several partners make a contract among each other concerning the common thing, and do so under equal conditions, and if they do not take anything away from the company as a whole, their claim is not lessened.

For the whole authority still applies to the company, and the members stand in the same relationship to each other. And I recognize that the authority applies to the company in the same way if the members have not given anything to a third person or taken anything away for themselves.

This has to be understood as referring to people who feel no more pain about loss than joy about gain, that is, people who are not harmed much by loss, or who can continue the game. For imposing this condition on them does not mean imposing a burden.

But so far, imposing the need to play nevertheless has meant imposing a burden because the mind is occupied without any profit, for our claim has not been increased.

So this is only favourable for those people for whom the occupation of the mind by the game equals the joy which they get from playing.

When there is only hope without fear, the occupation of the mind can in no way be regarded as a disadvantage. This happens if another person has contributed to the pool. If the hope is greater than the fear, taking up the game is right for a prudent man who has leisure and can continue.

But let us move away from considering what can increase or diminish the desire to play in a man. Instead, let us look at those who have already taken up the game or tend to play often; or let us posit that the game is a frequent affair, so that anyone can easily find a buyer of his expectation by auction sale when he wants to. In that case we shall argue as follows:

The hope is worth as much as the authority to have the thing.

The authority to have the thing in every outcome is the claim to the whole thing.

The authority to have the thing in some outcome compared to the authority to have the thing in every outcome is like the possibility of the one outcome to the possibility of all outcomes. If the outcomes are equally easy or equally possible, the authority to have the thing in one outcome compared to the authority to have the thing in every outcome is like the number 1 compared to the number of outcomes.

Let us suppose that the outcomes are equally easy. As the whole claim and the authority to have the thing in every outcome is the same, the authority to have the thing in one outcome compared to the whole claim is like the number 1 compared to the number of outcomes.

If the outcomes are equally easy, the authority to have the thing in one outcome is that part of the claim to the thing, or of the estimated value of the thing, which we get by setting it in proportion to the number of outcomes.[6]

The same will be understood if I imagine that I become the heir of the players: since all have the same right, it follows that I make as much profit by the death of one as by the death of another, and when I acquire the total by the death of all, it follows that I have received each one's stake by the death of all the individuals.

So as it has been shown that *we can be seen to have so much in credit as is the probability of having*, and that *we can be seen to lack so much of the thing as is the probability of losing* (for it was this about which I remember Roberval to have had doubts), we will solve the rest easily in the following way:

Theorems:

1) If there are several equally easy outcomes, and if I have the thing in one outcome, but none of the others, my hope will be worth that part of the thing that we get by setting it in relation to the number of outcomes.

Let the number of outcomes be n and the thing itself R;[7] the hope s will be $\frac{R}{n}$. This is exactly what we showed slightly earlier: if the outcomes are equally easy,

[6] *Aliquota portio* in the Latin text basically means 'proportion', hence my translation of *portio aliquota (...) pro* as 'that part (...) that we get by setting it in proportion to.'

[7] The symbols Leibniz uses are abbreviations of the Latin words, for example R = *res*, 'thing'. We keep Leibniz's symbols rather than taking abbreviations of the English equivalents.

the claim to having the thing in one outcome is that part of the thing's estimated value that we get by setting it in relation to the number of outcomes.

2) If there are several equally easy outcomes, and if I am going to have the thing in some of them, while in some others I am going to forfeit it, the estimated value of my hope will be the share of the thing which is such in relation to the whole thing as the number of outcomes that can be favourable for me compared to the number of all outcomes.

Thus $\frac{s}{R} = \frac{f}{n}$, that is, $s = \frac{f}{n}R$.[8] The demonstration is obvious: if there are several equal, favourable outcomes, the estimated value of my hope is a multiple of that estimated value which applies if there are several equal outcomes,[9] for the same is merely repeated several times. It should be noted that when I have said 'if there are several equally easy outcomes' so far, I have taken it that those several outcomes are all possible outcomes.

If all outcomes are equally easy, and if to each one of them some thing is assigned which I am going to have in this outcome, my hope will be that portion of the sum of things that we get by setting it in relation to the number of outcomes.

$s = \frac{A+B+Cetc}{n}$, for example $\frac{A+B+C}{3}$. The demonstration depends on theorem 1, for I can have A in only one single outcome, whereas I am going to forfeit it in the others. Consequently, the estimated value of my hope of getting A is $\frac{A}{n}$. In the same way, the estimated value of B is $\frac{B}{n}$, and so on. Thus, the estimated value of my entire hope is $\frac{A+Betc}{n}$.

The reasoning concerning the fear of losing is the same as that concerning the hope of having. For my fear of losing is other people's hope of having.

If hope and fear coincide in things that can be evaluated by price or a common measure, the final hope or fear will be the difference between the first hope and fear.

If, once the same conditions are posited, the first hope is greater than the first fear, the final hope will be the surplus of hope over fear, and the final fear is smaller than zero. The opposite is also true: if the first fear is greater, the final fear will be the surplus of fear over hope, and the final hope will be smaller than zero.

If of all events some give A, some others B, and the rest C, the total hope will be the sum of the individual things, each multiplied by the number of outcomes which can yield it, divided by the number of all possible outcomes.

[8] f is the number of favourable outcomes.

[9] This means: 'if there are several equal outcomes and only one of them is favourable.'

Thus, if the number of outcomes which can give A is α, the number of outcomes which can give B is β, and the number of outcomes which can give C is γ, and if the number of all outcomes is n, the hope s equals $\frac{\alpha A+\beta B+\gamma C}{n}$. The demonstration of this is obvious, for it is as if we were to posit as outcomes so many things as are all outcomes $\alpha+\beta+\gamma$, for example $A, M, N, B, P, Q, C, R, S$. s will then equal $\frac{A+M+N+B+P+Q+C+R+S}{n}$.

Let us now posit that A, M and N are equal, or $A = M = N$, and that the number of repetitions of A is α. αA will equal $A + M + N$. Similarly, if $B = P = Q$, and the number of repetitions of B is β, $B + P + Q$ will be the same as βB. And if $C = R = S$, and the number of repetitions of C is γ, $C + R + S$ will be the same as γC. Consequently, $s = \frac{\alpha A+\beta B+\gamma C}{n}$. *QED.*

If the number of possible outcomes is greater than that of the cases to which something has been assigned, the same will nevertheless have to be said as before.[10] Thus, it is unnecessary for $\alpha + \beta + \gamma$ to equal n.

For it is as if 0 had been assigned to the remaining outcomes to which nothing has been assigned. 0 can be added or deleted without negative effects, for example $s = \frac{\alpha A+\beta B+\gamma C+\delta 0}{n}$, having posited that $n = \alpha + \beta + \gamma + \delta$. This is the same as $s = \frac{\alpha A+\beta B+\gamma C}{n}$.

If two people play under the condition that the person who has won three times first has won completely, and if I have won twice, it can be asked what hope of winning completely I have, or what my hope is worth. It is obvious that if he also wins once[11] we will be equal. Let us imagine that everything necessarily has to be finished completely in six decisive rounds.[12] By decisive I mean rounds in which someone of us wins. Let us posit now that the rationale of the game is such that one of the two has to win or to lose in each round. Initially, before any of us wins, we are equal. Afterwards I win in one round. It is obvious that the other person has to win once as well for us to become equal again. But he has only five rounds, so his risk is much greater than if he still had more rounds. He wins in turn, and everything will be brought back to the state of equality. Moreover, it will be just as if the game came to such a state that it is exactly as if we had fixed from the outset that the person who wins in two rounds first is understood to have won the game.

[10]Leibniz is now considering null outcomes.

[11]That is, if instead of me winning twice I win once and he wins once.

[12]If the outcome of the game is decided as soon as one of the players has won three times, the maximum number of rounds necessary to reach a decision is five. The players may have a sixth round, but this sixth round will not change the result at all. In general, if there are y players, and if the outcome of the game is decided when one of them has won z times, the maximum number of rounds necessary to reach a decision is $(y \times (z-1)) + 1$.

This must be recognized if victories and losses always count in the same way in the game,[13] but it is different if one has to win in three consecutive rounds that are not interrupted by any loss.

But this does not matter now. It is enough that everything should come back to the state of equality. So in the second round we either return to equality, or to a situation in which the other needs two rounds for equality. Let us posit that I have won in the second round as well. Then that state of affairs comes about that I either win or that the matter goes back to the previous situation. My hope of winning the whole game is the other's of bringing it about that the previous state returns, in which he needed one round to equality and I two to victory. He needs three rounds for victory, I only two.[14]

Let it be fixed from the outset that the person who wins in two rounds first thereby wins the game, and that I win in one round. It is clear that by this victory I have brought it about that our hopes are unequal in the second round. For with the justification with which I hope for victory he merely hopes for equality. So with the justification with which I hope for the whole he hopes for a half. Therefore, the estimated value of my hope is twice that of his hope, and consequently two thirds of the thing are owed to me and one third to him,[15] even though he needs three successful rounds[16] for victory and I only one. For the number of rounds cannot be estimated because the values of the individual rounds are different or unequal.[17]

Nevertheless, there is a difficulty in what I have said, namely, 'with the justification with which I hope for the whole, he hopes for a half.' For the question does not appear to be what I hope to have, but what I hope to gain.[18] But the two of us can be seen to have something already, albeit something different. Let me have y and him x. Let the whole thing R be $y + x$. I hope to gain $R - y$, that is R minus

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[13]I have rendered *si ludus aequalitate pensatus numeratur* more freely.

[14]I only need one round if I have won twice. What Leibniz says is true after the first round. If Leibniz had written *erat* instead of *est* in this sentence, it would be correct.

[15]The obvious mistake is that the players' fears have not been considered. I hope to have R, but fear to get $\frac{R}{2}$ if the other player wins. I ought to get the value in between my hope and fear, that is $\frac{R+\frac{R}{2}}{2} = \frac{3}{4}R$. The other player hopes to get $\frac{R}{2}$, less than me, but his fear is also smaller than mine: he fears to get 0 if I win. So what he should get is $\frac{\frac{R}{2}+0}{2} = \frac{R}{4}$.

[16]Obviously, he has to win twice, not three times. Perhaps by 'successful rounds' Leibniz means 'rounds in which one of us wins, regardless of which one', and since I have won once, there will have been three rounds if the other person wins the game by succeeding in two rounds. 'Three successful rounds' would also be correct if one had to win not just twice, but in two more rounds than the opponent.

[17]By 'the number of rounds' Leibniz seems to mean 'the number of rounds required to win'.

[18]This will turn out to be false. The reason is that I do not really possess anything yet. If we look at what I gain, we wrongly presuppose that I already have something in my possession.

what I have, and that in turn is $y + x - y$, thus x. So I hope to gain x. He hopes to gain $\frac{R}{2} - x$, or $\frac{R}{2}$ minus what he has, thus $\frac{y+x}{2} - x$, or $\frac{y-x}{2}$.

Consequently, I hope to gain x, and he hopes to gain $\frac{y-x}{2}$. But what I can gain and what he can gain must be the same in this game, of course after that in which we are unequal has been deducted.[19] Thus, $x = \frac{y-x}{2}$, so $2x = y - x$ and $3x = y$. Now $y = R - x$, so $R = 4x$ and $x = \frac{R}{4}$. From this calculation something other than above would be found, namely that the hope is to be estimated in relation to the number of rounds which are necessary for victory. We are supposing, of course, that apart from the claim which he already has, each of the players has the same claim to obtaining new profit in the following round, or that the profit which each obtains in the second round is equal. But this is not yet certain. What is certain is that in the beginning each one has the hope of equal gain; after that it can happen that the person who has won also has also a greater hope of new gain than the other.

But perhaps there is an error in the calculation, so let us look at it again: when I won in the first round, I obtained at any rate something, which we shall now call y. What we still have to gamble on is consequently $R - y$. Of course the calculation shall be undertaken in such a way that what I have gained should be given to me in advance, and that then, in the second round, we should contend for the rest, and that in this way the same should finally come about as if no sum had been given in advance, but as if all had been continued according to the rules of the game.

Therefore, since the pool is $R - y$, my claim to the rest will be $\frac{R-y}{2}$. And if I win, I shall win the whole $R - y$, which becomes R if added to the y in my possession, but from there we do not learn anything.

Initially I have $\frac{R}{2}$. The first victory has the value y for me. Thus I have $\frac{R}{2} - z$. The first loss has the value z for the other person, so he has $\frac{R}{2} + y$.[20] Now $\frac{R}{2} - z + \frac{R}{2} + y = R$. So $z = y$. Consequently,[21] the winner has $\frac{R}{2} + y$ after the first round, the loser $\frac{R}{2} - y$. If I win in the second round I get v, but then I have the whole: so $\frac{R}{2} + y + v = R$, or $y + v = \frac{R}{2}$; the other person has nothing, for he has $\frac{R}{2} - y - v$, and $\frac{R}{2} - y - v = 0$. If I am beaten in the second round, then I lose x, or I obtain $-x$, while my opponent obtains x, and I have $\frac{R}{2} + y - x$, my opponent $\frac{R}{2} - y + x$. But in this way matters come back to equality, and as

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[19]This is wrong, as Leibniz says at the end of the paragraph. We merely have equal chances of winning the second round.

[20]In the preceding calculation, y and z are both negative.

[21]Now Leibniz uses only one variable, y, which is positive.

individual players we have only regained half the claim to the whole pool R; thus, $\frac{R}{2} + y - x$ or $\frac{R}{2} - y + x = \frac{R}{2}$, so $y - x = 0$ or $y = x$.

I posit that there are two equally easy[22] outcomes, one that I lose x, that is to say y, the other that I gain v. So my expectation of gaining what I do not have yet is $\frac{\mp y \pm v}{2}$ or half the difference between v and y. And this hope is what I have gained by winning in the first round. For I have gained nothing else apart from the hope of winning more easily than the other player in the second round. So $\frac{\mp y \pm v}{2}$[23] $= y$. So $2y = \mp y \pm v$. Hence $\mp$ must be $-$, and $\pm$ must be $+$. Otherwise y would be $-v$. So[24] $v - y = 2y$, that is $v = 3y$. But above we saw that $y + v = \frac{R}{2}$. So $4y = \frac{R}{2}$, hence $y = \frac{R}{8}$. Now the person who won the first game has $\frac{R}{2} + y$, so he has $\frac{R}{2} + \frac{R}{8}$ or $\frac{4R}{8} + \frac{1R}{8}R$, that is $\frac{5R}{8}$.[25]

But if we think from the beginning that by my first victory I have an expectation either of $\frac{1}{2}$ if I lose in the second round, or of 1 if I win, the value of my expectation obtained by my first victory will be $\frac{\frac{1}{2}+1}{2}$ or $\frac{3}{4}$.

This does not tally with our earlier statement; therefore, we have to examine very carefully if we should ask about the whole or about gain and loss.[26]

Initially I have an equal hope either of obtaining all by the outcome of the game, or of losing all, so I have $\frac{R+0}{2}$ to begin with. In the first round I gained something by my victory, which I shall call y. So I have gained the hope of acquiring everything. In other words, as I already have one half or $\frac{R}{2}$, and y, I have gained the hope of acquiring now $\frac{R}{2} - y$. Yet I have also got the fear of losing y. But that hope – it is the very quantity y – will be $y = \frac{\frac{R}{2} - y - y}{2}$, that is $2y = \frac{R}{2} - 2y$, or $y = \frac{R}{8}$.

But it can be seen that one can reply to this reasoning that y cannot be the hope in respect to itself, but has to be defined by something else. For it is said that I have acquired the hope of gaining $\frac{R}{2}$ minus this hope together with the fear of losing this hope itself. But it seems that that hope must not be defined by such a recourse to itself.[27] By imagining that the players have no power, but that merely hopes have been given, and that the person who won in the first round has the hope of gaining 1 or $\frac{1}{2}$, and that the person who lost in the first round that of gaining $\frac{1}{2}$ or 0, it is

[22]By this Leibniz means 'equally possible.'

[23]The edition reads $\frac{\mp y \pm v}{v}$, which is obviously wrong.

[24]That is, in reality.

[25]There is a mistake in the edition, which reads $\frac{R}{2} + \frac{R}{8}$ or $\frac{4R}{8} + \frac{5}{8}R$.

[26]We should of course ask about the whole. In the next paragraph, Leibniz looks at what happens if we ask about gain and loss. After that, he comes back to the whole.

[27]Leibniz is saying that an element to be defined may not be used in the definition itself.

obvious that the former person's hope will be $\frac{3}{4}$, the latter person's $\frac{1}{4}$. This is very true and has to apply here because it must be recognized that none of the players has anything more than these hopes; and if anyone wants to give him alternative hopes, which we defined above, he can do no better than to put the players into this situation.

If we evaluate the hope in the above way in terms of itself, we might see what comes about if the other person's hope is evaluated as well. So he has the prospects either of gaining y or of losing v, which is $3y$. Now $y = \frac{R}{8}$, and v will be $\frac{3R}{8}$. So his prospects are $\frac{y-v}{2}$, that is $-y$ or $-\frac{R}{8}$. In reality he has gained $-y$, that is, he has lost it. But that is irrelevant. It is enough that we have shown that the total hope ought to be estimated.[28]

Let it now be the case that one has to win in three rounds. Let the estimated value of the first victory be y, that of the second (y), and that of the third $((y))$. In the first round one obtains y, that is the intermediate value of $\frac{R}{2}$ and (y), that is, $y = \frac{R}{4} + \frac{(y)}{2}$. In the second round one obtains the intermediate value of y and $((y))$, that is, $(y) = \frac{y+((y))}{2}$. So $y = 2(y) - ((y)) = \frac{R}{4} + \frac{(y)}{2}$,[29] and so $6(y) = R + 4((y))$.[30] $(y) = \frac{y+((y))}{2}$. $((y)) = \frac{(y)+(((y)))}{2}$. $((((y)))) = \frac{(((y)))+(((((y)))))}{2}$. The values of the intermediate terms can always be found with the help of the first and the last element.

But when this has been understood correctly, it is obvious that it helps for the easier calculation of the arithmetic progression. Its nature is of such a kind that half the sum of two things equals the middle value. So we have to posit an arithmetic progression $\frac{R}{2}, y, (y), ((y)), (((y))), ..., R$. Its middle terms are as many as the number of rounds minus one, or [the whole progression is][31] one more than the number of rounds. The difference between $\frac{R}{2}$ and y will be found by dividing the difference between $\frac{R}{2}$ and R, that is to say, the difference between the first and the last member, which is $\frac{R}{2}$, by the number of middle terms. Here (in the table) there are five of them, so the difference between the first term, $\frac{R}{2}$, and the second term, y, will be $\frac{R}{10}$.

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[28]The reason is that we get an absurd result if we look at gain and loss rather than the whole: if we do this, the other person loses a part of R even if he wins the next round.

[29]$y = 2(y) - ((y))$ may not be immediately obvious, but follows from $(y) = \frac{y+((y))}{2}$ because from there it can be seen that $2(y) = y + ((y))$, and consequently $y = 2(y) - ((y))$.

[30]Above it was stated that $2(y) - ((y)) = \frac{R}{4} + \frac{(y)}{2}$. $\rightarrow 8(y) - 4((y)) = R + 2(y) \rightarrow 6(y) - 4((y)) = R \rightarrow 6(y) = R + 4((y))$. Note that I left out *seu y* in my translation because it seems to be a mistake.

[31]I have added this.

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