Leibniz on Estimating the Uncertain: An English Translation of *De incerti aestimatione* with Commentary

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Abstract

Leibniz's *De incerti aestimatione*, which contains his solution to the *division problem*, has not received much attention, let alone much appreciation. This is surprising because it is in this work that the definition of probability in terms of equally possible cases appears for the first time. The division problem is used to establish and test probability theory; it can be stated as follows: if two players agree to play a game in which one has to win a certain number of rounds in order to win the pool, but if they break the game off before either of them has won the required number of rounds, how should the pool be distributed?

Our article has two aims: it provides the readers with the first published English translation of *De incerti aestimatione*, and it also gives them a brief commentary that explains Leibniz's philosophical and mathematical concepts necessary in order to understand this work. The translation is as literal as possible throughout; it shows how Leibniz struggled at times to find a solution to the division problem and how he approached it from different angles. The commentary discusses Leibniz's views on four key concepts: fairness, hope, authority and possibility. The commentary then outlines how Leibniz attempted to solve the problem of division.

Commentary by James Cussens

1 Introduction

Leibniz's 1678 manuscript *De incerti aestimatione* (*DIA*) contains the first appearance of the 'Laplacian' definition of probability in terms of equally possible cases. This alone would be enough to establish the manuscript as a turning point in the philosophy of probability. However, in *DIA* Leibniz supplies far more than an abstract axiom of equipossibility. The key argument in *DIA* is that:

> probability is degree of possibility
> *probabilitas est gradus possibilitatis*

and it is this idea which is "the ultimate source of the Laplacian definition of probability" (Hacking, 1975).

The greater part of *DIA* concerns not philosophy, but a mathematical analysis of the *division problem*, the very problem that Chevalier de Méré posed to Pascal leading to the birth of the probability calculus. The proposed solution to the division problem in *DIA* is generally held not to constitute any advance over Huygens's earlier work (1657), which Leibniz must have seen (Biermann & Faak, 1957). The assertion that *DIA* "did not lead to new, fruitful results for the division problem" (Biermann & Faak, 1957) is representative of the general view, although Struve and Struve (1997) dissent. Indeed Hacking (1975, p.125) claims that *DIA* did not even aim to solve the division problem.

Once written, "*De incerti aestimatione* was buried among a mass of other papers inaccessible to [Leibniz's] contemporaries" (Schneider, 1981). Some extracts were later made available by Couturat (1903, p.569–571), who also briefly discussed it (Couturat, 1901). The article by Biermann and Faak (1957) contains the original Latin in full, together with a German commentary, which concentrates on mathematics. More recently, *DIA* has been discussed by a number of writers (Hacking, 1975; Schneider, 1981; Krüger, 1983; Parmentier, 1993; Crombie, 1994; Struve & Struve, 1997; Franklin, 2001). The translation contained in this article, which appears to be the first published full translation into English, is from the *Sämtliche Schriften und Briefe* (Leibniz, 1999).

2 Fairness

For Leibniz, estimation, including that of the uncertain, is a form of *judgement* "indissolubly linked to the Leibnizian conception of probability" (Parmentier, 1993).

> The estimated value of a thing is as high as each one's claim to it.
> *Aestimatio rei tanta est, quantum est jus cujusque in rem.*

Leibniz had a legal conception of probability which predated his education in mathematical probability (Hacking, 1975). Leibniz viewed probability as a quantified *right*: this is the key to understanding *DIA*. Most of *DIA* concerns fair games of chance and it is only natural that Leibniz is more concerned to investigate what makes a game fair than to simply assume the existence of fair games and then go on to deduce the mathematical consequences—this latter option being Huygens's approach. Leibniz begins *DIA* with his criterion for a fair game:

> A game is fair if there is the same proportion of hope to fear on either side.
> *Justus ludus est si spei et metus utrinque eadem ratio sit.*

Hope of winning is distinguished from fear of losing, and this distinction is maintained throughout *DIA*. As Biermann notes: "Only Jakob Bernoulli gave up this distinction, by explaining expressly, that by hope in games we understand not only the expectation of a favourable, but also an unfavourable result (Bernoulli, 1713)." Note that here it is enough for the ratio hope:fear to be constant for all players. Hope need not equal fear for fairness to obtain. Nevertheless, Leibniz restricts the rest of his analysis of fair games to the case where hope does equal fear.

Leibniz specifies what is required for an equal proportion of hope to fear:

> Axiom. If players do similar things in such a way that no distinction can be drawn between them, with the sole exception of the outcome, there is the same proportion of hope to fear.
> *Axioma. Si ludentes similia agunt ita ut nullum discrimen inter ipsos assignari possit, nisi quod in solo eventu consistat eadem spei metusque ratio est.*

Hacking has tentatively proposed that this axiom is an application of what Keynes (1921) called *the principle of indifference*; an epistemic mode of reasoning where if *we know of no reason* to expect one player to win rather than another we give them equal probabilities of winning. However, Leibniz's axiom gives a condition on *what is the case*, rather than what we know (or do not know): namely that "the players do similar things". This is an *aleatory* condition concerning what Popper would call *generating conditions*, not an epistemic one. The only thing we do not know is who will win. That "no distinction can be drawn between" the players is merely a necessary epistemic consequence of this objective state of affairs.

In most of this early section of *DIA* Leibniz describes fairness and/or equality of hope in terms of the objective conditions of the game rather than in terms of our knowledge of the game, providing evidence against Hacking's epistemic interpretation. Here is a typical example:

> If the pool is formed by common, equal contribution of the players, if each one is playing in the same way, and if for the same outcome the same prize or the same penalty is fixed, the game is fair.
> *Si sors communi ludentium contributione aequali formetur, et unusquisque eodem modo ludat, et eidem eventui idem praemiuim eademve poena statuatur, justus ludus est.*

However, the Axiom itself is supported by a metaphysical argument which lends much greater weight to Hacking's interpretation:

> This [namely the Axiom] can be demonstrated by metaphysics: where the appearances are the same, the same judgement can be formed about them
> *Potest demonstrari ex Metaphysicis, nam ubi quae apparent eadem sunt, idem de iis judicium formari potest*

So Leibniz provides two routes to equal judgements. We get there if players actually operate under the same conditions or if merely "appearances are the same". Leibniz implies that the latter metaphysical route is required to validate the former. This dual approach suggests that:

> Leibniz was probably confused and he almost certainly vacillated in his conception of probability. He sometimes leans to an epistemic notion, sometimes to an aleatory one. (Hacking, 1975)

Leibniz lacks the notion of an unknown expectation or probability, and it is this notion which yields a much more straightforward justification for ascribing equal expectations in fair games. Consider a fair game where there are two players and in each round each player throws the same die. The game stops when one player throws a six and the other does not, the six-throwing player thus winning. Let P_i be the unknown probability of throwing a six on the ith throw. It is enough to require that $P_i = P_{i+1}$ for all $i = 1, 3, 5 \dots$ (what we would now call a *stationarity condition*) to ensure that each player has probability $\frac{1}{2}$ of winning. From (assumed) equal unknown probabilities concerning throws of a die, known probability values concerning pairs of throws can be derived.

3 Hope

Leibniz always assumes that, in a fair game, each player contributes the same amount to the pool. The *value of hope* is then simply determined:

> Once the same conditions have been posited, each one's hope is worth as much as he contributed.
> *Iisdem positis uniuscujusque spes tantum valet quantum contribuit.*

So hope, a "thought[] about the future outcome", is a commodity with a determinable value: what we would call today *expectation*. This is possible since in *DIA* all hopes are (implicitly) *rational hopes*. Leibniz sometimes refers to how much a hope is worth and sometimes to the *estimated* value of hope: these are interchangeable terms for Leibniz.

Leibniz then makes a couple of obvious points. In a fair game, if the pool remains constant and the number of players increases then the chances of winning decrease but, since each player's stake must decrease, "the fear of losing is also smaller". Similarly, if the players' contributions remain constant but the number of players increases then "the greater the hope of profit is, [since the pool has increased] but the greater is also the fear of losing [since there are more competitors]."

Leibniz then makes his first link between hope in general and fair games:

> If I expect either A or B with the same hope, the estimated value of my hope will be half the sum of A and B. For it is exactly as if there were two players and A was awarded to one by the outcome, B to the other.
> *Si vel A vel B expectem aequali sperandi ratione erit aestimatio spei dimidium summae ex A et B. Perinde est enim ac si duo sint lusores et eventus uni A alteri B adjudicet.*

Note however, that this is no definition of expectation (and thus implicitly probability) in terms of fair games, but the reverse. If circumstances lead us to expect A and B equally, then we can interpret them in terms of some fictional fair game, but this is not a method to produce equally expected alternatives in the first place.

But Leibniz then goes on to give a condition which provides just such a method and, further, derives the estimated value of hope in the general case:

> To put it in a general way, if the various successful outcomes can be kept separate in an activity, the estimated value of my hope will be the sum of possible successes collected from all outcomes, divided by the number of outcomes.
> *Generaliter si diversos eventus utiles disjunctim habere possit negotium, spei aestimatio erit summa utilitatum possibilium ex omnibus eventibus collectarum, divisa per numerum eventuum.*

Leibniz is apparently generalising from fair games to any given "activity". Expectation is defined as the ratio of successes to all outcomes. There is no explicit symmetry condition (either aleatory or epistemic) on the outcomes—it is apparently enough that "the various successful outcomes can be kept separate". This appears to be an epistemic condition, providing further evidence that Leibniz vacillated between aleatory and epistemic conceptions of probability.

4 Authority and possibility

The basic conclusion so far is that, in a fair game, the value of hope (i.e. expected winnings) equals the amount staked. Leibniz continues by providing an alternative way in which we "can reach the same conclusion". This new argument starts by stating some of the key themes of *DIA*:

> Probability is the degree of possibility. Hope is the probability of having. Fear is the probability of losing. The estimated value of a thing is as high as each one's claim to it.
> *Probabilitas est gradus possibilitatis. Spes est probabilitas habendi. Metus est probabilitas amittendi. Aestimatio rei tanta est, quantum est jus cujusque in rem.*

Recall that hope is an expectation—a value not a probability—and this is also how *probability of having* should be interpreted. A later passage reinforces this point:

> we can be seen to have so much in credit as is the probability of having
> *tantum nos in bonis habere videri, quanta est habendi probabilitas*

Leibniz's second argument centres not on *hope* but on *authority*. Authority, for Leibniz, can be partial and partial authority is what probability is. This echoes Leibniz's 1665 Baccalaureate dissertation *De conditionibus*, which assesses the strength of conditional legal rights on a scale from 0 to 1 (Hacking, 1975, p.87–88).

Leibniz argues that equal conditions are not sufficient for each player to have an equal claim on the pool. Each must also suffer the same harm from losing and the same joy from winning. His analysis is thus restricted to people "who are not harmed much by loss" and "for whom the occupation of the mind by the game equals the joy which they get from playing." The interesting inference from this is that it would be just for a player who enjoys playing to have a lesser claim on the pool than one who did not enjoy playing. Similarly, rich players have a lesser claim than poor ones since, presumably, they are less harmed by loss. Leibniz's argument is a remarkable anticipation of modern decision theory where winning a game may have a different *utility* for different agents, and this utility need not be simply money.

Leibniz then turns to objective conditions which allow him to connect hope and authority:

> ...let us posit that the game is a frequent affair, so that anyone can easily find a buyer of his expectation by auction sale when he wants to. In that case we shall argue as follows: The hope is worth as much

> as the authority to have the thing.
> *... posito ludum commercium frequens esse, ita ut quis facile emtorem expectationis suae cum velit, per subhastationem invenire possit tunc ita ratiocinabimur: Spes tanti est, quanti est potestas habendi rem.*

Leibniz is careful to state that only if expectations are *liquid* in the financial sense can trade between hope and authority be effected. Crudely speaking, the connection between hope and authority is then hope $=$ authority $\times$ thing or in modern terms expectation $=$ probability $\times$ value. Leibniz then takes "a decisive step" (Schneider, 1981) and connects authority (i.e. probability) to *possibility* or *easiness*:

> If the outcomes are equally easy or equally possible the authority to have the thing in one outcome compared to the authority to have the thing in every outcome is like one compared to the number of outcomes.
> *Si eventus sint aeque faciles seu aeque possibiles potestas habendi rem in unum eventum est ad potestatem habendi rem in omnem eventum, ut unitas ad numerum eventuum.*

In fair games it is equally possible for each player to win since they play under the same conditions. Leibniz now generalises to any set of equally possible outcomes. The definition is not circular since it infers a legal claim from a physical state of affairs. The difficulty is to formulate general conditions guaranteeing that outcomes are equally possible in the first place. In *DIA* Leibniz only supplies the special case of fair games. A *general* theory of possibility is provided by Leibniz's metaphysical theory of possible worlds. Differing analyses connecting this theory to Leibniz's philosophy of probability are given by Hacking (1971, 1975), Wilson (1971) and Krüger (1983).

We have previously accused Leibniz of vacillating between an aleatory, objective view of probability and an epistemic one. Schneider (1981, 28n) argues that Hacking's view that Leibniz provides both aleatory and epistemic interpretations "... ascribes to Leibniz considerable insight in modern concepts of probability". Hacking (1975, p.126) himself warns that this analysis "may be an anachronism; it is too tempting to wish upon Leibniz distinctions clear to us and obscure to him".

In fact, Leibniz's juridical concept of probability is neither aleatory nor epistemic: "Legal concepts and techniques thus bestow on Leibniz's conception of probability a 'legal objectivity' that, by avoiding a priori probabilization and equiprobability, allowed him to escape the objective/subjective alternative" (Parmentier, 1993). The legal concept which permits such an avoidance is that of fairness.

Interestingly, as Franklin (2001, p.365) explains, Leibniz never succeeded in applying mathematical probability to the law. For example, when, in 1703, Bernoulli asked for legal examples where posterior probabilities could be applied, Leibniz could supply nothing. Leibniz thus represents "not only the coming together of legal and mathematical probability, but also their divergence" (Franklin, 2001).

5 The division problem

The division problem (also known as *the problem of points*) concerns games where there are two players (A and B), and where the first player to win k rounds (for some k) wins a pool of money to which A and B have previously contributed equally. Each player has an equal chance of winning any given round. If the game is interrupted at a point where A has won a rounds, and B has won b rounds the problem is how to divide the pool 'fairly' between A and B, as a function of a, b and k.

From a modern perspective the problem is simple. Compute the probability that A would have won, had the game continued, and give that proportion of the pool to A, the rest to B. This probability is $2^{-n}\sum_{i=0}^{k-b-1}\frac{n!}{i!(n-i)!}$ where $n = 2k-a-b-1$ is the maximum number of rounds left to play. This is Pascal's famous solution of 1654. Previous to this breakthrough, throughout the fifteenth century, there had been a number of different proposed solutions. Some doubted that the problem was soluble: in 1556 Tartaglia, despite producing a solution of his own, stated that "the resolution of such a question must be judicial rather than mathematical, so that in whatever way the division is made there will be cause for litigation" (Hacking, 1975, quoted on p. 51).

For Leibniz, the resolution was naturally both judicial and mathematical. His judicial reasoning did not always produce Pascal's answer, as his first proposed solution demonstrates:

> Let it be fixed from the outset that the person who wins in two rounds first thereby wins the game, and that I win in one round. ... with the justification with which I hope for victory he merely hopes for equality. So with the justification with which I hope for the whole he hopes for a half. Therefore, the estimated value of my hope is twice that of his hope, and consequently two thirds of the thing are owed to me and one third to him,
> *Sit ab initio pactum, ut qui prior duabus lusionibus vicisset, ludo vincat, ego vincam una lusione ... quo ego jure spero victoriam, hoc ille*

> *sperat aequalitatem, id est quo jure ego spero totum, hoc jure ille sperat dimidium, itaque est aestimatio spei meae spei illius dupla, ac proinde duo trientes rei mihi debentur, ipsi unus.*

The problem with this reasoning is that the amount owed to each party is taken to be proportional to hopes, not to the overall position of the players which includes both hope and fear.

Leibniz then produces a different solution based on the following argument:

> But what I can gain and what he can gain must be the same in this game
> *Debet autem id quod ego et quod ille lucrari potest hoc ludo idem esse*

Leibniz clarifies this using the same example as above. Let my current position be worth y and his x, where $x + y = R$, R being the total pool. Since I am one away from victory my hope is to get R in total and thus to *gain* $R - y = x$ in the next round. It is easy to show that he hopes to gain $\frac{y-x}{2}$. Leibniz's principle of equal gain dictates that $x = \frac{y-x}{2}$ from which it follows easily that $x = \frac{R}{4}, y = \frac{3R}{4}$, the right answer. Struve and Struve (1997) have shown that this principle of equal gain always produces Pascal's answer, but do not mention that Leibniz considered it.

Leibniz continues by providing a third, rather convoluted, alternative argument which considers hopes about winning other hopes and which Biermann has analysed. This third argument leads eventually to yet another value, $\frac{5R}{8}$, for Leibniz's running example. Interestingly, immediately after the effort of producing this incorrect solution, Leibniz provides a simple and correct fourth approach:

> But if we think from the beginning that by my first victory I have an expectation either of $\frac{1}{2}$ if I lose in the second round, or of 1 if I win, the value of my expectation obtained by my first victory will be $\frac{\frac{1}{2}+1}{2}$ or $\frac{3}{4}$.
> *Si vero ab initio cogitemus prima victoria me habere expectationem vel ad* $\frac{1}{2}$ *si vincar lusione secunda, vel ad* 1 *si vincam, erit valor expectationis prima victoria* $\frac{\frac{1}{2}+1}{2}$ *seu* $\frac{3}{4}$.

Here, unlike in the first argument, both hopes and fears are considered, leading to the correct answer. Leibniz returns later to this approach claiming it is "obvious" and "very true". *DIA* finishes with Leibniz incorrectly applying the method of arithmetic middles which Biermann has analysed in some detail.

In the mathematical section of *DIA*, as indeed to an extent in the philosophical section, Leibniz appears to be experimenting with different modes of reasoning and seeing where they lead. As Biermann puts it, although unfinished, Leibniz's work on the division problem is:

> ...of interest because the details of Leibniz' explanations give us insight into his 'mental workshop' and reveal the methods of working used by Leibniz.

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References

Bernoulli, J. (1713). *Ars conjectandi.* Basel.

Biermann, K.-R., & Faak, M. (1957). G. W. Leibniz' „De incerti aestimatione“. *Forschungen und Fortschritte*, *31*, 45–50.

Couturat, L. (1901). *La Logique de Leibniz d'après documents inédits.* Alcan, Paris. Also published by Georg Olms, Hildesheim in 1961.

Couturat, L. (1903). *Opuscules et fragments inédits de Leibniz: Extraits des manuscrits de la Bibliothèque royale de Hanovre.* Alcan, Paris. Available via `http://galica.bnf.fr`. Also published by Georg Olms, Hildesheim in 1966.

Crombie, A. C. (1994). *Styles of scientific thinking in the European tradition*, Vol. 2. Duckworth.

Franklin, J. (2001). *The Science of Conjecture: Evidence and Probability before Pascal.* The John Hopkins University Press, Baltimore.

Hacking, I. (1971). The Leibniz-Carnap program for inductive logic. *Journal of Philosophy*, *68*(19), 597–610.

Hacking, I. (1975). *The Emergence of Probability.* Cambridge University Press.

Huygens, C. (1657). De ratiociniis in aleae ludo. *Exercitationum mathematicarum*, *5*, 517–534. Frans van Schooten, Amsterdam.

Keynes, J. M. (1921). *A Treatise on Probability.* Macmillan, London.

Krüger, L. (1983). Probability in Leibniz: On the internal coherence of a dual concept. *Archiv für Geschichte der Philosophie*, *63*, 47–60.

Leibniz, G. W. (1999). De incerti aestimatione. In *Gottfried Wilhelm Leibniz: Sämtliche Schriften und Briefe*, Vol. 6 (Sechste Reihe, Vierter Band, Teil A), pp. 91–101. Akademie Verlag. Original written in September 1678.

Parmentier, M. (1993). Concepts juridiques et probabilistes chez Leibniz. *Revue d'Histoire des Sciences*, *46*, 439–485.

Schneider, I. (1981). Leibniz on the probable. In Dauben, J. W. (Ed.), *Mathematical Perspectives: Essays on Mathematics and Its Historical Development*, pp. 201–219. Academic Press.

Struve, H., & Struve, R. (1997). Leibniz als Wahrscheinlichkeitstheoretiker. *Studia Leibnitiana*, *29*(1), 112–122.

Wilson, M. (1971). Probability, propensity, and chance: Some doubts about the Hacking thesis. *Journal of Philosophy*, *68*(19), 610–617.