Leibniz: Geometry, Physics, and Idealism

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Abstract

Leibniz holds that nothing in nature strictly corresponds to any geometric curve or surface. Yet on Leibniz's view, physicists are usually able to ignore any such lack of correspondence and to investigate nature using geometric representations. The primary goal of this essay is to elucidate Leibniz's explanation of how physicists are able to investigate nature geometrically, focussing on two of his claims: (i) there can be things in nature which approximate geometric objects to within any given margin of error; (ii) the truths of geometry state laws by which the phenomena of nature are governed. A corollary of Leibniz's explanation is that physical bodies do have boundaries with which geometric surfaces can be compared to very high levels of precision. I argue that the existence of these physical boundaries is mind-independent to such an extent as to pose a significant challenge to idealist interpretations of Leibniz.

1. Introduction

From the 1680's and until the end of his life, Leibniz is a steadfast advocate of the view that nothing in nature corresponds strictly to any geometric object. No body has a "precise" or "definite" geometric shape, and the motions of bodies do not happen along geometric curves.[1,2] On the other hand, Leibniz does not appear to take this failure of correspondence between geometry and nature to have any serious methodological consequences for the practicing mathematical physicist. The physicist is free to represent and reason about bodies and their motions as though these do have some precise shape or curve using all the tools of geometry and the new calculus.[3] Indeed, we can see an example of this attitude in Leibniz's own scientific practice: in "An Essay on the Causes of Celestial Motions", Leibniz offers a demonstration that the trajectories of the planets of our solar system, such as Mars, are elliptical. Leibniz shows no concern in that text for his independent arguments that, strictly speaking, the orbit of Mars could not be an ellipse. Those arguments are simply ignored.[4]

The principal goal of this essay is to explain Leibniz's justification for holding that in certain ordinary mathematical and scientific contexts, one is justified in representing and reasoning about nature as though it contains "precise shapes"—for instance, as though it contains bodies and motions strictly corresponding to continuous geometric curves and surfaces—despite his own arguments that there can be no such "precise shapes" in nature. When Leibniz is confronted with this difficulty, the justification he tends to offer is that even if nothing in nature corresponds exactly to any geometric shape, there can be things in nature which approximate geometric shapes to within any specified margin of error. For example, Leibniz writes in 1679 that "[E]ven if straight lines and circles do not and cannot possibly exist in nature, it suffices nonetheless that there can exist figures which differ so little from straight lines and circles that the error be less than any given" (A6.4.159).[5] Part of my task will be to explain how and to what extent this kind of justification solves the original difficulty. In particular, I will argue that if he is correct, Leibniz manages to explain how in spite of all his arguments against precise shapes in nature, the *phenomena of nature* may still be just as if there were precise shapes.

Some interpreters of Leibniz, recently Timothy Crockett, contend that Leibniz's arguments against precise shapes in nature are at the same time arguments against the reality of extended physical objects *tout court*.[6] Such interpretations regard Leibniz's rejection of precise shapes in nature as a key step on the road to his mature idealism. The secondary goal of this essay is to make a case against idealist interpretations of Leibniz's arguments concerning precise shapes in nature. In particular, I will argue that it is a corollary of Leibniz's solution to the difficulty just considered that one can make sense of precise quantitative discrepancies between geometric objects and the shapes of natural objects. Thus even if physical objects do not have "precise shapes", they do have some shapes—or if one prefers one may speak here of some extensional properties—with which precise shapes can be compared to very high degrees of precision. Samuel Levey's suggestion is that the shapes actual things have according to Leibniz closely resemble what we would today recognize as fractal curves.[7] I will provide evidence for thinking that Levey's suggestion is compatible with Leibniz's view of the way nature approximates geometry.

The plan of the essay is as follows. In §2 I will offer a brief account of Leibniz's view that nothing in nature corresponds precisely to any geometric curve or shape, focussing primarily on those features of the physical world which are incompatible with the existence of "precise shapes". In §3 I will analyze Leibniz's justification for treating natural objects as though they have precise shapes, and in general for using mathematical techniques in the empirical sciences, even when nothing in nature strictly corresponds to any geometric object. Finally, in §4 I will use the preceding discussion to argue against those idealist readings of Leibniz according to which Leibniz's arguments against precise shapes are ultimately arguments against the reality of shape and extension *tout court*.

2. A World Without Precise Shapes

Leibniz offers a number of different arguments against the existence of precise shapes in nature; without individuating too finely, one may recognize at least four lines of argument: (1) an argument that precise shapes are ruled out by the actually infinite division of matter directly and without consideration of time (the so-called "synchronic" argument);[8] (2) an argument that precise shapes are ruled out by the infinite division of matter together with facts about how bodies and their motions are in perpetual flux over time (the so-called "diachronic" argument);[9] (3) an argument from sense perception that any hypothesis about an object's shape would be ruled out by a better, more complex hypothesis if the object were seen under more powerful magnification;[10] (4) an argument that precise shapes in nature would amount to barren or uncultivated parts of the universe and are therefore incompatible with God's wisdom.[11] My focus will be to explain how the features of the world revealed by the first argument rule out the existence of precise shapes in nature.[12]

Leibniz presents an account of the fundamental physics of the world which he understands to preclude the existence of precise shapes in nature. He repeatedly asserts that the absence of precise shapes in nature is due to the fact that bodies, or the parts of matter which make up bodies, are actually infinitely divided.[13] To understand Leibniz's position it will be helpful to consider first the account of the division of matter, then the way in which the division of matter precludes precise shapes in nature.

The actually infinite division of matter comes about as a consequence of the way bodies move through a plenum, *i.e.*, through spaces which are entirely filled by other bodies. Leibniz endorses the argument from Descartes' *Principles of Philosophy* according to which a fluid moving through a hose or container with different diameters along its length will have speeds in inverse proportion to the diameters.[14] If the hose's diameter is always changing, then the speed at any two locations—no matter how close they are to each other—will have to be slightly different. To accommodate the different diameters and fill the container, the bodies composing the fluid will break up into ever smaller pieces. Descartes refrains from describing this fracturing as actually infinite, preferring to describe it as "indefinite" and incomprehensible to us.[15] Leibniz argues that the fracturing should be understood as actually infinite (A6.3.553–556, Ar, pp. 181–187). Thus while Leibniz endorses the core of Descartes' argument, he gives a slightly different interpretation of its results. Also, because on Leibniz's view the condition of the fluid moving through the hose with differing diameters is representative of bodies' general situation in our universe—the universe being entirely filled by bodies, each of which is moving through unequal places in the plenum—Leibniz takes the argument to show that all bodies are actually infinitely divided.

We now want to see how it follows from the fact that bodies are actually infinitely divided that bodies do not have any precise or definite shape. This is a contentious issue among Leibniz interpreters. Leibniz tends not to fill out the details of the argument, and there are several different proposals for how to do so. In this section I will follow the presentation given by Levey in his paper "Leibniz on precise shapes and the corporeal world". In §4 I will consider another suggestion made by Timothy Crockett.

Suppose for the sake of contradiction that there is a body whose surface is a precise, geometrically definable curve. Such a surface must be continuous in the mathematical sense. However, by the argument of the last few paragraphs, in any interval of the supposed surface of the body, there will be infinitely many distinct parts marked off as such by their slightly different motions in comparison with their neighbors. The surface of the original larger body is in fact composed of the surfaces of the smaller parts. These parts are contiguous, *i.e.*, they touch each other and there is no space in between them. But they are not continuous, since they do not share one and the same boundary but rather have distinct surfaces where they touch. It follows that there are actually infinitely many discontinuities over any interval we consider. Since all precise, geometrically definable surfaces were assumed to be continuous, it follows that the body does not have a precise, geometrically definable surface over any interval. Of course, it would be no use to move down a level and consider only the surface of a part of the original body, since we can start with any part of the body and run the *reductio* once again.

Leibniz draws similar conclusions also about times and motions:

> [I]t will be worthwhile to consider the harmony of matter, time and motion. Accordingly I am of the following opinion: there is no portion of matter that is not actually divided up into further parts…Similarly there is no part of time in which some change or motion does not happen to any part or point of

> a body. And so no motion stays the same through any space or time however small. . . (A6.3.565–566, Ar, p. 209)

It follows that uniform motion or uniform acceleration are not to be found in nature, either. Rather, over any stretch of time a body's motion is subject to infinitely many variations as it is being battered by the bodies surrounding it. The trajectories of bodies through space, then, will not be precise geometric curves either.[16]

Even on Leibniz's conception of them, physical bodies and their motions do have what we can recognize as a mathematical structure, taking "mathematical structure" in a fairly broad sense. Bodies have extrema—*viz.*, surfaces and points. Bodies also have parts, and these parts stand in part-whole relations of arbitrarily high complexity. However, it is important to see that the structure of bodies, motions, and other physically "continuous" aspects of nature is not the same as the structure of geometric objects such as curves or surfaces. As Leibniz explains in a letter to the Electress Sophie of Hanover of 1705:

> The fact is that matter, the evolution of things, and finally every genuine composite, is a discrete quantity, but that space, time, mathematical motion, intension or the continual increase that is conceived in speed or other qualities. . . is a continuous and undetermined quantity in itself, or one indifferent to the parts that can be taken from it and which are actually taken in nature. The mass of bodies is actually divided in a determinate manner and there is nothing exactly continuous in it; but space or the perfect continuity which is in the idea marks only an undetermined possibility of dividing it as one will. In matter and in actual realities the whole is a result of the parts, but in ideas or possibles. . . the whole is prior to the divisions (G7.562).[17]

Leibniz is in effect proposing new definitions for the terms "mathematically continuous" and "physically continuous" according to which continuity amounts to something very different in each case. A mathematically continuous quantity is a whole prior to any possible division of it into parts. All consistent ways of partitioning a continuous quantity are equally possible; we may choose to divide it into parts in whatever way we like. A physically continuous quantity is actually divided into parts which are prior to the whole quantity and together constitute it. How the quantity is divided into parts is determined by the physical facts, in particular by facts about motion. Points or surfaces only exist in the physically continuous quantity when there are parts which have those points or surfaces as boundaries. Moreover, the parts of the physically continuous quantity have their own separate boundaries which merely touch one another. This is the sense in which they are not really continuous, for if they were really continuous, two parts which were next to each other would not have separate boundaries. The following shows in a particularly clear way how the structure of the mathematically and physically continuous are different from each other: In a physically continuous quantity such as a body, there can be distinct points p and p' whose distance from one another is zero. For instance, these points may lie on the surfaces of two parts which are touching each other. In a mathematically continuous quantity, if p and p' are at zero distance from one another, then $p = p'$.[18]

In some contexts, and especially when issues in the foundations of geometry and physics are concerned, Leibniz stresses the difference between the mathematically continuous and the physically continuous. In other contexts, especially if his remarks are aimed at practicing mathematicians or physicists, Leibniz plays down the importance of the distinction. The main reason the practicing physicist may ignore the distinction is that on Leibniz's view, physically continuous things or processes can approximate mathematical continuity to any given margin of error. In the next section I will consider how such approximations are possible and how they form a part of Leibniz's justification of geometric methods in physics.

3. An Error Less than Any Given

Leibniz's rejection of precise shapes in nature raises a worry about his justification for applying geometry to physics. Such applications involve representing and reasoning about bodies and their motions as though they correspond to geometric curves or surfaces. But Leibniz argues on many occasions that such correspondences never hold in full strictness. Leibniz therefore needs a justification for the applicability of geometry to physics which is compatible with the failure of any strict correspondence between nature and the objects of geometry.

Leibniz is aware of this worry, and his response to it is multifaceted. In the following discussion it will be helpful to distinguish between two explanatory goals Leibniz might be aiming at in justifying the applicability of geometry to physics. One goal would be to explain how geometric truths either can or do count as laws of *the phenomena of nature*, or how nature appears to us, despite the fact that there are no precise geometric shapes in nature. This would help to explain why we are justified in holding it to be true of the phenomena of nature that the area of any ellipse (say) is equal to the product of π, its semi-major axis a, and its semi-minor axis b, and therefore why we are entitled to appeal to geometric truths in physics.

For purposes of abbreviation, I will call this the goal of explaining how geometric truths govern the phenomena. A different goal would be to explain how we are justified in taking any particular phenomenon, for instance, the trajectory of Mars as it appears to us, to be approximated by some geometric curve, or how we are justified in reasoning about the trajectory of Mars by means of the approximation. I will call this the goal of explaining the existence and legitimacy of geometric approximations. In this essay, I will consider only Leibniz's explanation of how geometric truths govern the phenomena, leaving the issue of the existence and legitimacy of geometric approximations for another occasion.[19]

It is clear enough from a number of texts that according to Leibniz, geometric truths do govern the phenomena. In 1695 Leibniz writes that "Number and line are not chimerical things. . . for they are relations that contain eternal truths, by which the phenomena of nature are ruled" (G4.491–492, AG, pp. 146–147); and in a letter to De Volder written on January 19, 1706, Leibniz claims:

> The science of continua, *i.e.* of possibles, contains eternal truths that are never violated by actual phenomena, since the difference [between real and ideal] is always less than any assignable given difference (G2.282–283, AG, pp. 185–186).[20]

In this respect mathematical truths are like metaphysical truths, both of which Leibniz refers to as "eternal laws" to which the appearances conform (G2.275, AG, p. 181). Because on several occasions Leibniz insists that geometric truths govern the phenomena only a few lines after arguing against the existence of precise shapes in nature, there is a burden on Leibniz to explain how both of these could hold at once.

Leibniz confronts the worry about how geometric truths *can be* laws of the phenomena of nature as well as the related but distinct worry about whether they *are* laws of the phenomena. Regarding the former, it is clear that geometric truths could not be laws of the phenomena if those phenomena were to "violate" geometry, a possibility Leibniz considers in his reply to Bayle's encyclopedia entry on Rorarius (cf. G4.568, L, p. 583). I am unaware of any text, including the reply to Bayle, in which Leibniz spells out what it would be for the phenomena to violate geometry. One plausible suggestion is that Leibniz is entertaining the kind of natural "counterexample" to geometry familiar from the Aristotelian tradition: material spheres that fail to touch a plane at a point, material triangles whose angles do not sum up to two right angles, *etc.*[21] But perhaps Leibniz has something else in mind.

At any event, Leibniz's response to the worry that the phenomena might violate geometric truths is to insist that conformity with geometric truths is a criterion of reality in phenomena. In the same reply to Bayle, Leibniz writes:

> Yet the actual phenomena of nature are arranged, and must be, in such a way that nothing ever happens which violates the law of continuity... or any of the other most exact rules of mathematics... Actual things cannot escape [mathematics'] rules. In fact, we can say that the reality of phenomena, which distinguishes them from dreams, consists in this fact (G4.568–569, L, p. 583).

Also, in the letter to De Volder discussed above, the fuller context is as follows:

> The science of continua, *i.e.* of possibles, contains eternal truths that are never violated by actual phenomena, since the difference [between real and ideal] is always less than any assignable given difference. And we don't have, nor should we hope for, any mark of reality in phenomena, but the fact that they agree with one another and with eternal truths (G2.282–283, AG, pp. 185–186).

These considerations foreclose any possibility that natural phenomena should violate or contain counterexamples to geometric truths. Any course of experience we might have which appeared to violate geometric truths should be rejected as unreal, as analogous to a dream, precisely because it does not cohere with geometric truth. Moreover, coherence with the eternal truths of mathematics and metaphysics is what makes an experience an experience of something real, as opposed to a dream or a hallucination.

The next element in Leibniz's explanation of how mathematical truths can govern the phenomena is an insistence that it is at least possible for there to be natural objects which approximate geometric objects arbitrarily closely. That Leibniz believes such approximations are possible comes out clearly in a text entitled "De Organo sive de Arte Magna Cogitandi", where Leibniz writes the following:

> For even if straight lines and circles do not and cannot possibly exist in nature, it suffices nonetheless that there can exist figures which differ so little from straight lines and circles that the error be less than any given. That is sufficient for the certainty of demonstration as well as practice. That figures of this kind can exist, however, is easily demonstrated, if only this one thing is admitted, namely that some lines are given.[22]

I interpret the syntax of Leibniz's claim to be the following: For any margin of error ϵ and any circle C, it is possible that there is a natural object N such that the difference between C and N is less than ϵ. A similar claim holds at least for lines.

However, the context of the quotation is the question of how one can construct the various geometric curves given some particular curves, such as circles and lines, as primitives. If one can use natural approximations to circles and lines in place of true circles and lines to construct the remaining curves, I presume one can also obtain arbitrarily good natural approximations of those remaining curves (by means of compass and straightedge constructions). In that case, we would have it that in general, for any margin of error ϵ and any geometric curve Γ constructible with compass and straightedge, it is possible that there is a natural object N such that the difference between N and Γ is less than ϵ.[23] Leibniz is not explicit about what is meant by the difference or error in this claim. I assume he means there is some way of superimposing the natural object onto the curve so that the distance from the curve to the natural object is always less than ϵ. This is how I will understand the difference or error between two curves in the remainder of this essay.[24]

If the difference between a geometric curve and the shape of a physical object is small enough, a human observer will never be able to distinguish the one from the other. That there are such limitations on human observers is something Leibniz emphasizes at various points in his career. For example, in an early work entitled "The Theory of Abstract Motion", Leibniz writes:

> [S]ensation cannot discriminate whether some body is a continuous or contiguous unit, or a heap of many discontiguous ones separated by gaps; whether parts are wholly at rest, or redound on themselves by an insensible motion; whether an angle of intersection is very slightly oblique, or exactly a right angle; whether the angle of contact is made at a point, or a line or surface. . . (A6.2.273, Ar, p. 343).[25]

In this text, Leibniz stresses that if from the physical theory he is proposing "no sensible error disturbs our reasons", then it "suffices for the phenomena" (*loc. cit.*). Leibniz makes similar remarks in defending the technique of approximating curves by large collections of polygons as one does in integral calculus. For when doing calculus one also argues that the difference between two quantities, namely the given curve and some corresponding approximation of it by polygons, can be made less than any given quantity. In such cases any error between the original curve and the approximation can be rendered completely "insensible".[26]

In summary, Leibniz's solution to the problem of how geometric truths can be laws of the phenomena of nature is as follows. Even though there are no precise shapes in nature, it is possible for things in nature to differ so little from precise shapes that this difference is beyond the limits of the acuity of human sensation. As far as the phenomena are concerned, things can therefore appear to have perfect shapes. If the difference between the shape of the natural object and the perfect geometric shape is sufficiently small, no sensible error can arise from conflating the two.

Now it does not follow from the fact that they *can* govern the phenomena that they *do in fact* govern them. Or in other words, it does not follow from the fact that there can be natural objects which approximate geometric objects to within any given margin of error that there are in fact such natural objects.[27] If no natural objects do suitably approximate the geometric ones, then the truths of geometry would at best amount to empty or vacuous laws. Leibniz seems to reject this possibility by insisting that eternal truths such as those of geometry *do* govern the phenomena, and that the difference between real and ideal *is* less than any given (as in the 1706 letter to De Volder cited above). However, he is far less explicit about which geometric objects are approximated by natural ones, and I am not aware of any place Leibniz takes this question on in a systematic way.

The one case Leibniz singles out for separate discussion is that of mathematical continuity itself. Leibniz writes in his reply to Varignon that "[O]ne can say in general that though continuity is something ideal and there is never anything in nature with perfectly uniform parts, the real, in turn, never ceases to be governed by the ideal and the abstract" (GM4.93, L, p. 544). There is both in this text and other texts the suggestion that the laws of the ideal, or of the mathematically continuous, hold also for the real, though I do not discern any explicit argument about which laws hold for both and which do not.[28] Earlier I gave as an example a law over which the two kinds of continua differ, namely the law that if the distance from point p to point p' is 0, then $p = p'$. Leibniz is aware of this discrepancy between the mathematically and the physically continuous, though it is not entirely clear what his response to the discrepancy is.[29]

A reasonably straightforward argument can be made, however, which I think does give some content to the idea that mathematical continuity and physical continuity differ from each other by less than any given amount. First, recall earlier that Leibniz said sense perception cannot distinguish "whether some body is a continuous or a contiguous unit" (A6.2.273, Ar, p. 343). This provides reason to think that on Leibniz's view, sense perception cannot distinguish whether there is only one point at a single location, or perhaps two or more. So one may identify contiguous points with each other without causing any error a human being can sense.[30] Now, for the time being taking physically contiguous points to be one and the same point, the second task is to see how physical continuity approximates mathematical continuity. The task is made difficult by the fact that Leibniz's usual definitions of mathematical and physical continuity are complicated and involve notions of possibility and determinacy. But on at least on occasion, Leibniz is willing to use a simpler criterion of mathematical continuity, as when he writes to Des Bosses that "when points are situated in such a way that there are no two points between which there is no midpoint, then, by that very fact, we have a continuous extension" (G2.515, AG, pp. 201–202). Now in the case of physically continuous quantities, points only exist as boundaries of bodies (or motions), so a midpoint between two points will only exist in the case that there happens to be a body (or motion) containing a boundary at that location. Nonetheless, given any small interval $\pm\epsilon$ around any location, we are guaranteed that there are points inside the interval since "there is no portion of matter that is not actually divided up into further parts" (A6.3.565, Ar, p. 209).[31] Therefore even if the physically continuous quantity does not always contain the midpoint for any pair of points in it, the physically continuous quantity does contain points within ϵ of the midpoints. I believe this is a reasonably straightforward sense in which the difference between physical and mathematical continuity is less than any given.[32] Since ϵ can always be chosen to be far finer than the level of acuity of human sensation, it follows that the physically continuous quantity will be indiscernible from the mathematically continuous quantity as far as all appearances are concerned.

While Leibniz stresses that mathematical and physical continuity differ by less than any given amount, he does not explicitly tell us which geometric objects are arbitrarily well approximated by natural objects. The claim Leibniz makes in the 1706 letter to De Volder suggests that such approximations are common. Recall Leibniz's statement: "The science of continua…contains eternal truths that are never violated by phenomena, since the difference [between real and ideal] is less than any assignable difference". "The science of continua" refers to geometry, and the difference between real and ideal presumably includes the difference between continuous mathematical objects and actual physical ones. Leibniz's statement therefore encourages a liberal view on the extent to which nature approximates geometry. In a similar spirit, Leibniz makes the remark about nature that "the more one knows her, the more geometric one finds her" (G3.54, L, pp. 352–353). Strictly speaking, however, none of this provides us with an argument or a precise description of which geometric curves or surfaces have arbitrarily good approximations in the physical world—or even merely very good approximations. This may not be a bad consequence, however; Leibniz can leave the matter open to observation and scientific investigation.[33]

4. Phenomenal and Worldly Aspects of Shape

In "The fluid plenum: Leibniz on surfaces and the individuation of bodies", Timothy Crockett develops an interpretation of Leibniz according to which the arguments against precise shapes in nature ultimately support the view that "the world, as it is in itself, does not contain genuinely extended things".[34] This means, in the first instance, that the world as it is in itself does not contain extended bodies with metaphysically determinate boundaries. But on Crockett's interpretation, the arguments against precise shapes also ultimately provide evidence against the reality of extended matter as well.[35] Crockett's interpretation is therefore in line with earlier interpretations of Leibniz which see the arguments against precise shapes in nature as providing Leibniz with support for idealism and an idealist analysis of body.[36] In the present section, I will argue that Leibniz's explanation of how geometric truths can be laws of the phenomena of nature poses a difficult challenge for Crockett's proposal about how to understand the arguments against precise shapes. In particular, Leibniz's explanation presupposes that there is a fact of the matter concerning the error or discrepancy between a mathematical curve and the shape of a body no matter how small the margin of error. It similarly presupposes a fact of the matter, no matter how small the error margin, about the discrepancy between physical as opposed to mathematical continuity. I will argue that the existence of discrepancies of the kinds just described requires that bodies have boundaries determinate to within any margin of error, and also that matter be extended. My arguments will therefore provide support for the view Crockett dubs "surface realism" and attributes to Levey.[37]

Let us return to the argument against precise shapes discussed in §2 which begins with the premise that matter is actually infinitely divided. As I presented it (following Levey), the argument also assumes for purposes of *reductio* the existence of a body whose surface is a precise, geometrically definable curve. I believe all the parties to the current debate on Leibniz on shape, including Crockett, Levey and myself, agree that so far as Leibniz's account of the phenomena is concerned, it does often seem to human observers that the bodies around them have shapes which are definite and geometrically definable—at the least from a distance and without the benefit of microscopes.[38] However, Crockett argues that the appearance of bodies with definite boundaries is ultimately shown to be "wholly phenomenal", *i.e.*, that determinate surfaces "only exist in perception or imagination".[39] Because he finds difficulties for Leibniz in the very idea of a body with a metaphysically determinate surface, Crockett does not analyze the argument against precise shapes as taking metaphysically determinate surfaces for granted (even for *reductio*).[40] Rather, on Crockett's interpretation the main assumptions of Leibniz's argument are the infinite division of matter together with a particular account, grounded in Leibniz's texts, of the individuation of bodies.

Briefly summarized, the argument goes as follows: If extended matter is infinitely divided, then all bodies in the universe are in fact fluid. Since Leibniz rejects the existence of vacuums, it follows that on his view the entire universe is filled with fluid. Some parts of the fluid evidently have more cohesiveness than others, although no part of the universe is perfectly hard or perfectly fluid, either. What makes something *one body* rather than many is the cohesiveness of its parts, where this is understood to mean their common motion or endeavor relative to other parts of the universe.[41] However, cohesiveness or common motion is a matter of degree. If we consider any given surface as a candidate for the boundary for a body, we will find fluids outside the surface moving together with parts in the interior; we will also find fluids inside the surface that do not have a motion in common with other parts in the interior. But then the surface fails to pick out a unique body, and since the candidate boundary was suitably arbitrary it follows in general that no surface could pick out a body. As Crockett puts it, "there is no fact about the world in virtue of which determinate boundaries among bodies exist".[42] Hence the apparent boundaries between bodies familiar from our everyday experience must exist only in our perception, not in the world. As a corollary to the main argument, it follows that there cannot be extended matter, either. This is because for extended matter to exist, matter must in some way be divided into parts which bear spatial relations to each other.[43] But the main argument has shown that the division of matter into parts cannot be made sense of, in effect because the boundaries of these parts cannot be made sense of. Hence Leibniz's entire argument can be taken to as a refutation of the claim that extended matter exists.[44]

I wish to argue against Crockett's reconstruction of Leibniz's argument on the grounds that it renders the conclusions of the argument inconsistent with Leibniz's explanation of how the truths of geometry can be laws of the phenomena of nature. I do not dispute that the premises of the argument, as Crockett presents them, are claims Leibniz would have endorsed at some time or other in his career. Nonetheless, I do not believe Crockett's reconstruction of the argument represents Leibniz's own settled view of what the argument accomplishes. This is chiefly because when one looks at the texts from the later parts of Leibniz's career in which Leibniz announces the absence of precise shapes in nature as a well established conclusion, nearby one often finds the explanation of how geometric truths can nonetheless be laws of the phenomena of nature—as if Leibniz is trying to forestall a misunderstanding.[45] Hence I take it to be important to interpret Leibniz's thesis that there are no precise shapes in nature compatibly with his explanation of how geometric truths can be laws of the phenomena of nature.

To see the inconsistency between Crockett's proposal and Leibniz's explanation of how geometric truths can be laws of the phenomena of nature, note first that if Crockett is correct, it follows not just that no body in nature actually has a determinate boundary, but also that it is physically impossible for a body to have a determinate boundary.[46] As against this, Leibniz's explanation relies on the fact that for any given margin of error and given geometric shape, it is possible for a body to exist in nature, the boundary of which differs from the geometric shape by less than the margin of error.[47] For there to be such determinate discrepancies between the boundary and the shape in turn presuppose that the bodies in question have boundaries that are determinate to within any margin of error, no matter how fine. For instance, suppose the boundaries of a body were allowed to be vague, leaving a collection of equally good candidate boundaries which never depart from each other by more than a nanometer. Then there would be no facts about the discrepancy between the shape of such a body and geometric shapes that hold with the margin of error of a picometer (a thousandth of a nanometer). Thus even if one wanted to hold that on Leibniz's view all physically possible bodies lack determinate boundaries, nonetheless the degree of vagueness or indeterminacy must be less than any finite assignable quantity.

Recall also Leibniz's explanation of how geometric truths manage to be non-vacuous with respect to the phenomena of nature. If my interpretation is correct, the non-vacuity of geometric truths requires that there actually be shapes in nature such that the difference between those shapes and some corresponding geometric curves is smaller than the acuity of our sense perception would allow us to detect. It would therefore follow from the non-vacuity of geometric truths that bodies have boundaries determinate enough that the difference of those bodies and some corresponding geometric curves is less than a margin of error determined by the acuity of our sensation. This, too, is incompatible with there being no fact of the matter about the boundaries of any body, though it is perhaps compatible with some larger degree of vagueness.

It appears to follow from the argument two paragraphs ago that extended matter is at least possible, since the bodies Leibniz is asserting to be possible are surely extended and presumably also made of matter. From the fact that Leibniz also seems to countenance natural bodies with shapes that are phenomenally indistinguishable from geometric shapes, or even natural bodies that closely approximate geometric shapes, it would seem to follow that extended matter is actual. Nonetheless, I think one gets a stronger argument for the actual existence of extended matter by considering Leibniz's statements comparing physical and mathematical continuity directly rather than by focussing on his statements about shape. The passages in the letters to De Volder and Varignon discussed in §3 indicated that physical continuity, while being importantly and strictly different from mathematical continuity, is nonetheless approximated by mathematical continuity to within any margin of error. But being spatially mathematically continuous is paradigmatically what it is to be an extended thing, so that if something can be approximated to within any margin of error by spatial mathematical continuity, that thing is surely extended.

Crockett is skeptical that there could be extended matter for Leibniz largely because he doubts that sense can be made of divisions or boundaries in matter. Crockett assumes that for Leibniz, what it is for matter to be extended is for it to consist of parts that are spatially related to each other. If one cannot make sense of boundaries between parts, one cannot make sense of the parts themselves or relations between parts, either. But if the account I proposed in §3 of the way physical continuity approximates mathematical continuity is correct, then Leibniz believes that for any location in a body and any margin of error ϵ, there is a division or boundary which is closer to the location than ϵ. So not only are there some facts about where divisions in matter exist, there are facts that hold to within any margin of error, no matter how small.

In light of the argument so far, I believe it should be granted that for Leibniz, extended matter and extended bodies with highly determinate boundaries are physically real. If that is correct, then the argument against precise shapes in nature induces a distinction between two classes of surfaces, the "ideal", "mathematical" surfaces which form the subject matter of geometry, and the "real", "physical" surfaces that correspond to the boundaries of bodies. Leibniz's argument attempts to show that the two classes of surfaces contain no surface in common. It does not attempt to show that extended bodies aren't physically real. Even granting this much, I believe there is *prima facie* a promising interpretive strategy left open to one who wishes to interpret Leibniz as an idealist. Such an interpreter may argue that being physically real is compatible with being wholly phenomenal and mind-dependent. Thus even though there is an important distinction between mathematical and physical surfaces from the point of view of the foundations of physics, from the point of view of philosophy the physical surfaces are every bit as phenomenal as the mathematical ones. In the remainder of this section, I wish to argue against this way of interpreting Leibniz as an idealist. Briefly put, my argument is that according to Leibniz, so far are physical surfaces from being wholly phenomenal that they are in fact not a part of human experience at all. This lends support to the view that the reality of physical surfaces is mind-independent at least insofar as it does not depend on the character of human sense perception or imagination.

Consider the following passage from Leibniz's 1705 letter to the Electress Sophie:

> There are. . . divisions and actual variations in the masses of existing bodies, to whatever limits one should go. It is our imperfection and the defect of our senses that makes us conceive physical things as mathematical beings, in which there is some undetermined thing. And one can demonstrate that there is no line or figure in nature which gives exactly and keeps uniformly through the least space and time the properties of a straight line or circle or something else whose definition can be comprehended by a finite mind. . . [N]ature cannot, and the divine wisdom does not wish to, trace exactly these figures of limited essence which presuppose something determined and consequently imperfect in the works of God. However, they are found in the phenomena, or in the objects of our limited minds: our senses do not recognize and our understanding conceals an infinity of little inequalities which nevertheless do not prevent the perfect regularity of the work of God, although a finite creature could not comprehend it (G7.563).[48]

Leibniz here draws the distinction between mathematical and physical surfaces along the lines just described. Mathematical surfaces are simpler, having "limited essences" and definitions finite minds such as ours can comprehend. When bodies and their motions appear to us, their shapes appear to us as mathematical surfaces and their trajectories appear as mathematical curves. But the appearances are misleading: nothing in nature or in the works of God corresponds precisely to mathematical curves and surfaces. The boundaries and trajectories of bodies are infinitely complex, they are beyond our comprehension, and they are not part of our sensory experience of the world. So described, it is hard to see how these boundaries could be "phenomenal" or the contributions of our sensory faculties. Rather, the physical boundaries exist in nature and are part of what God does, though they are hidden from us by the workings of our minds.[49]

The overall thrust of this section has been to argue against idealist readings of Leibniz's arguments that there are no precise shapes in nature on the grounds that such readings are not faithful to the ways in which physical surfaces are approximated by mathematical ones. In particular, I have argued that physical surfaces and divisions are according to Leibniz real in a sense that contrasts with "ideal"; that their existence is independent of the character of human sensory faculties and imagination; that they are determinate to within margins of error set by the acuity of human sensation; that it is physically possible for them to be determinate to within any finite, non-zero margin of error; that they are the work of nature or God rather than man. It may nonetheless be possible to square these views with the idealism ordinarily attributed to the mature Leibniz, *viz.*, a fundamental metaphysics according to which ultimately there are only simple mind-like substances together with their perceptions and appetites. My present point is only that in the context of his more comprehensive account of geometry and physical boundaries, Leibniz's arguments against precise shapes in nature provide at best weak support for the mature idealism. Recognition of the fact that the simple geometric shapes we attribute to bodies are not really properties of those bodies—the underlying physical reality being far more complex—*may* help to prepare us for Leibniz's more radical metaphysical proposal. On the other hand, rendering the account of that underlying physical reality consistent with the traditional idealist interpretation of Leibniz is not a straightforward task.

One might worry, however, that the surface realist is no better off than the traditional idealist interpreter of Leibniz; and the surface realist *would be* no better off if he could give no account of physical surfaces which are distinct from yet arbitrarily well approximated by geometric surfaces. In fact there is a surface realist proposal regarding the character of physical boundaries: In previous work, Levey has suggested that Leibniz's conception of physical surfaces anticipates the twentieth century notion of a fractal. Levey has offered the Koch curve as a particularly apt model for the kind of surfaces Leibniz takes real bodies to have.[50] One can show that Levey's suggestion is compatible with the facts about approximation I have been stressing. In particular, it can be shown that for any real number ϵ greater than zero, there is a continuous and everywhere differentiable curve which approximates the Koch curve to within ϵ.[51] This provides further evidence that the surface realist's conception of physical boundaries is on the right track.[52,53]

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Notes

1. In addition to the abbreviated names for Leibniz's works standardly used in this journal, I will use "Tentamen" as an abbreviation for G.W. Leibniz "An Essay on the Causes of Celestial Motions" in D. Bertoloni Meli (*Trans.*) *Equivalence and Priority: Newton vs. Leibniz* (Oxford: Clarendon Press, 1993), pp. 126–142. I will follow the common practice of citing the Akademie edition of Leibniz's works by series, volume, and page number, so that A6.4.159 refers to series 6, volume 4, p. 159. Similarly, I will cite the Gerhardt editions by volume number and page, so that G7.563 refers to volume 7, p. 563. Whenever possible, I will give citations in the Akademie edition (A). When this is not possible, I will give citations from Gerhardt (either G or GM), the formerly standard edition of Leibniz's works. When translations into English are also available, I will cite an English translation as well.
2. For example, writing in 1686, Leibniz remarks that "no determinate shape can be assigned to any body, nor is a precisely straight line, or circle or any other assignable shape of any body, found in the nature of things" (A6.4.1622, Ar, p. 315). Later, in his 1702 reply to Bayle, Leibniz reaffirms the view: "It is true that perfectly uniform change, such as the mathematical idea of motion,

is never found in nature any more than are actual figures which possess in full rigour the properties which we learn in geometry" (G4.568, L, p. 583).

3. See for example Leibniz's discussion of the issue in his reply to Bayle, *ibid.*, *loc. cit.*
4. Cf. (Tentamen), *passim*.
5. This is my translation. I will give the fuller context of this statement in §3; the original Latin text appears in n. 22.
6. See Timothy Crockett "Leibniz on shape and the Cartesian conception of body" in A. Nelson (*Ed.*) *A Companion to Rationalism* (Malden, MA: Blackwell Publishing, 2005), pp. 262–281, and Timothy Crockett "The fluid plenum: Leibniz on surfaces and the individuation of body" *British Journal for the History of Philosophy 17*:4 (2009), pp. 735–767.
7. See Samuel Levey "The interval of motion in Leibniz's *Pacidius Philalethi*" *Nous 37*:3 (2003), pp. 371–416, and Samuel Levey "Leibniz on precise shapes and the corporeal world" in D. Rutherford and J.A. Cover (*Eds.*) *Leibniz: Nature and Freedom* (Oxford: Oxford University Press, 2005), pp. 69–94.
8. Leibniz sometimes writes as if the actually infinite division of matter directly implies that there are no precise shapes in nature, as when he says to Arnauld: "...because of the actual subdivision of parts, there is no definite and precise shape in bodies" (G2.97–98, AG, p. 80). There has been considerable interpretive effort spent to understand the inference from infinite division to the exclusion of precise shapes in nature. For recent commentary, see Samuel Levey, "Leibniz on precise shapes and the corporeal world", as well as another paper by Levey, "*Dans les corps il n'y a point de figure parfaite*: Leibniz on time, change and corporeal substance" in M. Mugnai & E. Pasini (*Eds.*) *Corporeal Substances and the Labyrinth of the Continuum in Leibniz* (Stuttgart: Franz Steiner Verlag, forthcoming). See also the two papers by Timothy Crockett cited earlier. Robert Adams discusses this argument in *Leibniz: Determinist, Theist, Idealist* (Oxford: Oxford University Press, 1994), pp. 229–232 and employs the distinction between the synchronic and diachronic arguments from infinite division.
9. This argument occurs in a text entitled "There is no Perfect Shape in Bodies"; cf. (A6.4.1613–1614, Ar, pp. 297–299). For an extensive discussion of the argument, see Samuel Levey "*Dans les corps il n'y a point de figure parfaite*: Leibniz on time, change and corporeal substance".

10. Cf. Leibniz's letter to the Electress Sophie of October 31, 1705 (G7.562–563). Leibniz's letter to Arnauld in October of 1687 is also suggestive of this line of argument. For the letter, see (G2.119); for Robert Adams's discussion of it, see *Leibniz: Determinist, Theist, Idealist*, pp. 229–230.
11. See the letter to the Electress Sophie already cited (G7.561–565).
12. Why focus on this argument? The (synchronic) argument from infinite division merits special attention because it directly concerns the relationship between geometric curves and the physical world, and also because it appears to have played a central role in Leibniz's thought about the issue (see n. 8).
13. See for example Leibniz to Arnauld at G2.119, also "A Specimen of Discoveries" at (A6.4.1622, Ar, p. 315).
14. The argument occurs in Book II, Propositions 33 and 34. See René Descartes (C. Adam & P. Tannery, *Eds.*) *Oeuvres de Descartes* (Paris: Vrin, 1964–1976), Vol. VIIIa pp. 57–60.
15. Descartes, *ibid.*, Vol. VIIIa pp. 59–60.
16. It is important to distinguish the question whether trajectories correspond strictly to geometric curves from the question whether trajectories are indeterminate or vague. There is textual evidence that Leibniz does not take trajectories to be indeterminate, *e.g.*: "[T]he motion of any point whatever in the world takes place in a line strictly determined in nature [*d'une nature determinée*]. . . " (G4.558, L, p. 576).
17. This text is taken from an unpublished translation by Donald Rutherford of Leibniz's letter to Sophie of October 31, 1705.
18. Beeley writes that from his early works through his later career, Leibniz "more or less consistently employs a model at the core of nature so to speak which has its origins in an essentially mathematical concept"; cf. Philip Beeley "Mathematics and nature in Leibniz's early philosophy" in S. Brown (*Ed.*) *The Young Leibniz and his Philosophy (1646–1676)* (Dordrecht: Kluwer Academic Publishers, 1999), p. 138. I believe Beeley's statement is partly correct, but also partly misleading. The physical world has a great deal of structure according to Leibniz, but it is important to see that this structure is not the same as any geometric or mathematical structure. Nonetheless, the fact that the physical world has such a rich structure will help to explain how the physical world can so well approximate geometric structures. This in turn will help explain the applicability of geometry to physics. See §3.

19. I refer the eager reader to my Ph.D. dissertation, *Investigations into the Applicability of Geometry* (Harvard University, 2011), especially §4.2.2. As of this writing, the work is readily accessible online (ProQuest/UMI publication number 3462775).
20. The remark in brackets was added by the translators. From the context, it is reasonably apparent that Leibniz is talking about the difference between the real and the ideal.
21. For Averroës on this issue, see comment 8 of his commentary to Aristotle's *Metaphysics* B in Aristotle *Aristotelis Stagiritae Omnia Quae Extant Opera* (Venice: Iunta, 1550–1552), Vol. 8 p. 22 *et verso*. Benedict Pereira is an example of a Scholastic philosopher who pressed the issue of natural "counterexamples" to geometry quite hard. See Benedict Pereira *De Communibus Omnium Rerum Naturalium Principiis & Affectionibus* (Venice: Apud Andream Muschium, 1586), pp. 375–376. I discuss these philosophers and the question of how to understand the apparent natural counterexamples to geometry in Chapter 1 of my dissertation, especially in §§1.1–1.2.
22. This is my translation of the passage. Here is the original: "Nam etiamsi non darentur in natura nec dari possent rectae ac circuli, sufficiet tamen dari posse figuras, quae a rectis et circularibus tam parum absint, ut error sit minor quolibet dato. Quod satis est ad certitudinem demonstrationis pariter et usus. Posse autem dari hujusmodi figuras non difficulter demonstratur, modo admittatur hoc unum, aliquas dari lineas" (A6.4.159).
23. Geometry in Leibniz's time (and well before it) investigated curves which are not constructible using a compass and straightedge. I know of no reason why Leibniz would not countenance the physical possibility of arbitrarily good natural approximations of the then known nonconstructible curves as well as the constructible ones—at least in the case of continuous curves such as the cycloid. Inasmuch as doing so helps Leibniz secure the applicability of the theories of nonconstructible curves, it would seem to strengthen his position to admit the physical possibility of natural approximations of nonconstructible curves. But this issue is not addressed in the quotation discussed here, and I am unaware of any text in which Leibniz explicitly addresses the issue.
24. It should be noted that the process of comparing a geometric curve to a natural object is not trivial in the Leibnizian context where geometric objects and the physical world are structurally dissimilar. It appears to be a consequence of Leibniz's view that the geometric and the physical do not even have the same

topological properties: as I discussed in the preceding section, Leibniz countenances the existence of distinct physical points lying at zero distance from one another. Nonetheless, it is clear that Leibniz takes such comparisons to be possible and takes there to be quantitatively precise facts about the extent of the difference between a geometric and a physical object. I suspect that such comparisons require one first to identify all physical points at zero distance from one another, so that the space in which the objects are compared is geometric space. See the discussion of continuity later in this section.

25. When Leibniz speaks of the "angle of contact" here, he is presumably speaking of the contact between a sphere and a plane tangent to it.
26. See for example (A2.1.53), cited by Beeley in "Mathematics and nature in Leibniz's early philosophy", p. 140. Beeley's article contains numerous references to passages in which Leibniz argues that very small differences are either insensible or make no difference to the phenomena.
27. In other words still, we want to know for which geometric curves Γ we have the following claim: For any margin of error ϵ and any geometric curve Γ, there exists some natural object N such that the difference between N and Γ is less than ϵ.
28. See also (G4.568–569, L, p. 583).
29. Cf. (A6.3.563–564, Ar, p. 205).
30. There is one difficulty with this suggestion. Leibniz does at least once give a sensible criterion for whether bodies are continuous or merely contiguous, namely whether there is sensible cohesion between them. For example, given a spherical body on a flat body, one may tell whether these are continuous or merely contiguous as according to whether one can push the spherical body around without resistance. See (A6.3.537, Ar, p. 149). This contradicts Leibniz's earlier claim that sense perception cannot discern the difference between continuity and contiguity. I am unsure whether this is genuinely a change of mind or an inconsistency. I am inclined to think the strongest position for Leibniz is to insist that sense perception cannot distinguish contiguous points and therefore to abandon any sensible criterion for distinguishing continuity from contiguity.
31. Or, one might also say, since the parts are divided into further parts *ad infinitum*.
32. Richard Arthur appears to be aware of the sort of argument I have just rehearsed. He writes in his introduction to *The Labyrinth of the Continuum* that "precisely because of this unending internal division, matter corresponds arbitrarily

closely to the ideal of continuity. . . ” (Ar, p. lxxiii). See Arthur’s discussion of Leibniz’s mature view of the continuum at (Ar, pp. lxxii–lxxiii).

33. The exception here is continuity: it is *a priori* in roughly our sense of the word that physical processes are continuous. See Leibniz’s response to Malebranche for the sense of continuity in question at (G3.51–55, L, pp. 351–354). Earlier I gave some of the texts which explain why the continuity of physical processes is *a priori*: anything in conflict with it would be rejected as unreal.
34. Timothy Crockett “The fluid plenum: Leibniz on surfaces and the individuation of bodies”, p. 736.
35. *Ibid.*, pp. 750–751.
36. Cf. Robert Merrihew Adams *Leibniz: Determinist, Theist, Idealist*, esp. Chapter 9.
37. Cf. Samuel Levey “Leibniz on precise shapes and the corporeal world”.
38. In the case of Crockett, see “The fluid plenum”, p. 756 for the textual evidence. It is also worth pointing out that Leibniz himself does not hesitate to describe ordinary cases in which an observer sees something as circular or as having some other geometric shape. See, for example, the letter to Foucher at (G1.370, L, p. 152), and the 1705 letter to the Electress Sophie at (G7.563). I will discuss the latter text in more detail in a few paragraphs.
39. Timothy Crockett “The fluid plenum”, p. 750.
40. I assume that in talking of metaphysically determinate surfaces, Crockett is talking about non-vague surfaces, the existence and character of which is suitably mind-independent. The respects in which physical surfaces do or do not depend on minds will be an ongoing issue in this section. I take it that at a minimum, it would follow on Crockett’s view that if physical surfaces are merely apparent and are contributed by human sensory faculties, then they are mind-dependent.
41. Timothy Crockett “The fluid plenum”, pp. 752–754.
42. *Ibid.*, p. 755
43. *Ibid.*, p. 736 n. 6.
44. *Ibid.*, p. 759.
45. Here I have especially in mind the 1702 reply to Bayle, the 1702 letter to Varignon, and the 1705 letter to the Electress Sophie. These appear at (G4.554–571, L, pp. 574–585), (GM4.91–95, L, pp. 542–544), and (G7.558–565), respectively.
46. I believe both Crockett and I can grant that on Leibniz’s view it is metaphysically possible for there to be bodies with determinate boundaries, since God

might have created atoms which are perfectly hard, and these atoms would have had determinate boundaries. See the letter to Arnauld at (G2.119). For God to do so would contradict God's wisdom, however (cf. GM3.565, AG, p. 171). Nonetheless, given facts about motion in the actual world, and given Crockett's preferred account of what makes something one body, Crockett's argument, if successful, shows that bodies with determinate boundaries are physically impossible.

47. See the text of "De Organo" (A6.4.159) and the discussion in §3.
48. This is an unpublished translation by Donald Rutherford.
49. Another text by Leibniz which supports the interpretation I am offering appears in "Infinite Numbers" of 1676 (A6.3.498–499, Ar, pp. 89–91). The interpretation of "Infinite Numbers" is somewhat more difficult, however, owing to the fact that Leibniz goes back and forth over the question whether we have sensory consciousness of the complexities of physical shapes or not. I believe that even as a matter of interpreting "Infinite Numbers", the view can be sustained that for Leibniz, the infinitely complex physical shapes are not part of human experience. See Levey's discussion of this passage in "Leibniz on precise shapes and the corporeal world", pp. 78–83.
50. See Samuel Levey "The interval of motion in Leibniz's *Pacidius Philalethi*", esp. §6.
51. For a proof of this claim, see Appendix A of my doctoral dissertation, *Investigations into the Applicability of Geometry*.
52. This essay won the Leibniz Society of North America's essay competition for 2011.
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